# Ch: RELATIONS AND FUNCTIONS

## Key Concepts

1. A relation R between two non empty sets A and B is a subset of their Cartesian Product $A \times B$. If A = B then relation R on A is a subset of $A \times A$

2. If (a, b) belongs to R, then a is related to b, and written as a R b If (a, b) does not belongs to R then a $\not R$ b.

3. Let R be a relation from A to B.
Then Domain of $R \subset A$ and Range of $R \subset B$ co domain is either set B or any of its superset or subset containing range of R

4. A relation R in a set A is called **empty** relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

5. A relation R in a set A is called **universal** relation, if each element of A is related to every element of A, i.e., $R = A \times A$.

6. A relation R in a set A is called

   a. **Reflexive**, if $(a, a) \in R$, for every $a \in A$,

   b. **Symmetric**, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

   c. **Transitive**, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, or all $a_1, a_2, a_3 \in A$.

7. A relation R in a set A is said to be an **equivalence relation** if R is reflexive, symmetric and transitive.

8. The empty relation R on a non-empty set *X* (i.e. a R b is never true) is not an equivalence relation, because although it is vacuously symmetric and transitive, it is not reflexive (except when *X* is also empty)

9. Given an arbitrary equivalence relation R in a set X, R divides X into mutually disjoint subsets $S_i$ called partitions or subdivisions of X satisfying:

   - All elements of $S_i$ are related to each other, for all i
   - No element of $S_i$ is related to $S_j$ ,if $i \neq j$
   - $\bigcup_{i=1}^{n} S_j = X$ and $S_i \cap S_j = \phi$, if $i \neq j$

- The subsets $S_j$ are called Equivalence classes.

10. A function from a non empty set A to another non empty set B is a correspondence or a rule which associates every element of A to a unique element of B written as
$f:A \to B$ s.t $f(x) = y$ for all $x \in A$, $y \in B$. All functions are relations but converse is not true.

11. If $f: A \to B$ is a function then set A is the domain, set B is co-domain and set $\{f(x): x \in A\}$ is the range of f. Range is a subset of codomain.

12. $f: A \to B$ is one-to-one if

    For all $x, y \in A$ $f(x) = f(y) \Rightarrow x = y$ or $x \neq y \Rightarrow f(x) \neq f(y)$

    A one- one function is known as injection or an Injective Function.

    Otherwise, f is called many-one.

13. $f: A \to B$ is an onto function ,if for each $b \in B$ there is atleast one $a \in A$ such that $f(a) = b$

    i.e if every element in B is the image of some element in A, f is onto.

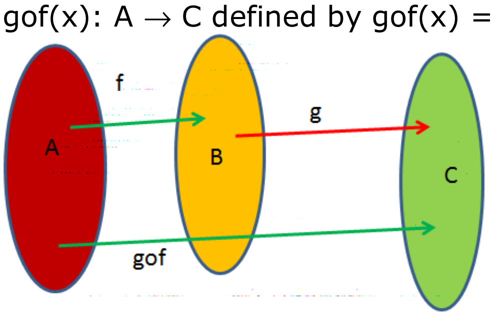
14. A function which is both one-one and onto is called a bijective function or a bijection.

15. For an onto function range = co-domain.

16. A one – one function defined from a finite set to itself is always onto but if the set is infinite then it is not the case.

17. Let $f: A \to B$ and $g: B \to C$ be two functions. Then the composition of f and g, denoted by gof is defined as the function $gof: A \to C$ given by

    $gof(x): A \to C$ defined by $gof(x) = g(f(x))\ \forall\ x \in A$




Composition of f and g is written as gof and not fog
gof is defined if the range of $f \subseteq$ domain of f and fog is defined if range of $g \subseteq$ domain of f

18. Composition of functions is not commutative in general

$fog(x) \neq gof(x)$.Composition is associative

If f: X→Y, g: Y→Z and h: Z→S are functions then

ho(g o f)=(h o g)of

19. A function f: X → Y is defined to be invertible, if there exists a function

g : Y → X such that gof = $I_X$ and fog = $I_Y$. The function g is called the inverse of f and is denoted by $f^{-1}$

20. If f is invertible, then f must be one-one and onto and conversely, if f is one- one and onto, then f must be invertible.

21. If f:A → B and g: B →C are one-one and onto then gof: A → C is also one-one and onto. But If g o f is one –one then only f is one –one g may or may not be one-one. If g o f is onto then g is onto f may or may not be onto.

22. Let f: X → Y and g: Y → Z be two invertible functions. Then gof is also

Invertible with $(gof)^{-1} = f^{-1} o\, g^{-1}$.

23. If f: R→ R is invertible,

f(x)=y, then $f^{-1}(y)=x$ and $(f^{-1})^{-1}$ is the function f itself.

24. A binary operation * on a set A is a function from A X A to A.

25. Addition, subtraction and multiplication are binary operations on R, the set of real numbers. Division is not binary on R, however, division is a binary operation on R-{0}, the set of non-zero real numbers

26.A binary operation ∗ on the set X is called commutative, if a∗b= b∗a, for every a,b∈ X

27.A binary operation ∗ on the set X is called associative, if a∗(b*c) =(a*b)*c, for every a, b, c∈ X

28.An element *e* ∈ *A* is called an **identity** of *A* with respect to *, if for each *a* ∈ *A*, *a* * *e* = *a* = *e* * *a*.
The identity element of (A, *) if it exists, is **unique**.

29.Given a binary operation ∗ from A× A →A, with the identity element e in A, an element a∈ A is said to be invertible with respect to the operation ∗, if there exists an element b in A such that a∗b=e= b∗a, then b is called the inverse of a and is denoted by $a^{-1}$.

30.If the operation table is symmetric about the diagonal line then, the operation is commutative.

| * | a | b | c | d |
|---|---|---|---|---|
| a | a | b | c | d |
| b | b | c | d | a |
| c | c | d | a | b |
| d | d | a | b | c |

The operation * is commutative.

31. Addition '+' and multiplication '·' on N, the set of natural numbers are binary operations But subtraction '–' and division are not since (4, 5) = 4 - 5 = -1$\notin$ *N* and 4/5 =.8 $\notin$ *N*

## Ch: INVERSE TRIGONOMETRY

### Key Concepts

1. Inverse trigonometric functions map real numbers back to angles.
2. Inverse of sine function denoted by $\sin^{-1}$ or arc sin(x) is defined on [-1,1] and range could be any of the intervals $\left[\frac{-3\pi}{2},\frac{-\pi}{2}\right],\left[\frac{-\pi}{2},\frac{\pi}{2}\right],\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$.
3. The branch of $\sin^{-1}$ function with range $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ is the principal branch.

   So $\sin^{-1}$: $[-1,1]\rightarrow\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$
4. The graph of $\sin^{-1}$ x is obtained from the graph of sine x by interchanging the x and y axes
5. Graph of the inverse function is the mirror image (i.e reflection) of the original function along the line y = x.
6. Inverse of cosine function denoted by $\cos^{-1}$ or arc cos(x) is defined in [-1,1] and range could be any of the intervals $[-\pi,0]$, $[0,\pi]$,$[\pi,2\pi]$.

   So,$\cos^{-1}$: $[-1,1]\rightarrow[0,\pi]$.
7. The branch of $\tan^{-1}$ function with range $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ is the principal branch. So $\tan^{-1}$: R $\rightarrow\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$.

8. The principal branch of $cosec^{-1}$ x is $\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$-{0}.

$cosec^{-1}$ x :R-(-1,1) $\rightarrow \left[\frac{-\pi}{2},\frac{\pi}{2}\right]$-{0}.

9. The principal branch of $sec^{-1}$ x is [0,π]-{ $\frac{\pi}{2}$ }.

$sec^{-1}$ x :R-(-1,1) →[0,π]-{ $\frac{\pi}{2}$ }.

10. $cot^{-1}$ is defined as a function with domain R and range as any of the intervals (-π,0), (0,π),(π,2π). The principal branch is (0,π)

So $cot^{-1}$ : R → (0,π)

11.The value of an inverse trigonometric function which lies in the range of principal branch is called the principal value of the inverse trigonometric functions.

**Key Formulae:**

1.Domain and range of Various inverse trigonometric Functions

| Functions | Domain | Range (Principal Value Branches) |
|---|---|---|
| $y = \sin^{-1} x$ | [−1, 1] | $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ |
| $y = \cos^{-1} x$ | [−1, 1] | $\left[0,\pi\right]$ |
| $y = cosec^{-1} x$ | R − (−1,1) | $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ |
| $y = \sec^{-1} x$ | R − (−1, 1) | $\left[0,\pi\right]-\left\{\frac{\pi}{2}\right\}$ |
| $y = \tan^{-1} x$ | R | $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ |
| $y = \cot^{-1} x$ | R | $\left(0,\pi\right)$ |

2. **Self Adjusting property**

$\sin(\sin^{-1}x)=x$ ; $\sin^{-1}(\sin x) = x$

$\cos(\cos^{-1} x)=x; \cos^{-1}(\cos x)=x$

$\tan(\tan^{-1} x)=x; \tan^{-1}(\tan x)=x$

Holds for all other five trigonometric ratios as well.

3. **Reciprocal Relations**

$\sin^{-1}\left(\frac{1}{x}\right)=\operatorname{cosec}^{-1} x$, $x \ge 1$ or $x \le -1$

$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, x \ge 1$ or $x \le -1$

$\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x, x > 0$

**4. Even and Odd Functions**

(i) $\sin^{-1}(-x) = -\sin^{-1}(x), \; x \in [-1,1]$

(ii) $\tan^{-1}(-x) = -\tan^{-1}(x), \; x \in R$

(iii) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x, |x| \ge 1$

(iv) $\cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1,1]$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \ge 1$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x, x \in R$

**5. Complementary Relations**

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1,1]$

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in R$

(iii) $\operatorname{cosec}^{-1}x + \sec^{-1} x = \frac{\pi}{2}, |x| \ge 1$

### 6. Sum and Difference Formuale

(i) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), xy < 1$

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right), xy > -1$

(iii) $\sin^{-1}x + \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} + y\sqrt{1-x^2}]$

(iv) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$

(v) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$

(vi) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}[xy + \sqrt{1-x^2}\sqrt{1-y^2}]$

(vii) $\cot^{-1}x + \cot^{-1}y = \cot^{-1}\left(\frac{xy-1}{x+y}\right)$

(viii) $\cot^{-1}x - \cot^{-1}y = \cot^{-1}\left(\frac{xy+1}{y-x}\right)$

**7. Double Angle Formuale**

(i) $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}, |x| \le 1$

(ii) $2\tan^{-1}x = \cos^{-1}\frac{1-x^2}{1+x^2}, x \ge 0$

(iii) $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, -1 < x < 1$

(iv) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), \frac{-1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(v) $2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2}), \frac{1}{\sqrt{2}} \le x \le 1$

**8. Conversion Properties**

(i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\frac{x}{x\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

$= \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) $\tan^{-1}x = \sin^{-1} = \frac{x}{\sqrt{1-x^2}}$

$= \cos^{-1}\frac{x}{\sqrt{1+x^2}} = \sec^2\sqrt{1+x^2}$

$= \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}$

*Properties are valid only on the values of x for which the inverse functions are defined.*