5 - 6. INDEFINITE INTEGRATION

5-6. INDEFINITE INTEGRATION

(1) Theorem 1 : If f and g are differentiable functions over (a,b) and $f'(x) = g'(x)\ \forall\ x \in (a,b)$ then $f(x) = g(x) + c\ \forall\ x \in (a,b)$ where c is an arbitrary constant.

Proof: Let $h(x) = f(x) - g(x),\quad x \in (a,b)$.
If $x_1, x_2 \in (a,b),\ x_1 < x_2$ then h is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).
Also by Mean Value Theorem,
$h(x_1) - h(x_2) = h'(c)(x_1 - x_2)$ where $c \in (x_1, x_2)$
Thus $c \in (a,b)$.

$\therefore\ h'(c) = f'(c) - g'(c) = 0$

$\therefore\ h(x_1) = h(x_2),\ \forall\ x_1, x_2 \in (a,b)$

$\therefore\ f(x_1) - g(x_1) = f(x_2) - g(x_2),\ \forall\ x_1, x_2 \in (a,b)$

$\therefore\ f - g$ is a constant function on (a,b).

$\therefore\ f(x) - g(x) = c$, where $c \in R$ is a constant.

$\therefore\ f(x) = g(x) + c\ \forall\ x \in (a,b)$.

(2) Theorem 2 : If f and g are integrable over interval $I \subset R$, then
$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

Proof: $\frac{d}{dx}\left[\int f(x)\,dx + \int g(x)\,dx\right] = \frac{d}{dx}\int f(x)dx + \frac{d}{dx}\int g(x)dx$
(Rule of derivatives)

$= f(x) + g(x)$
(by definition of integration)

$\therefore\ \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

In general, if $f_1, f_2 \cdots f_n$ are integrable over an interval, then $\int [f_1(x) + f_2(x) + \cdots + f_n(x)]\,dx = \int f_1(x)\,dx + \int f_2(x)\,dx + \cdots + \int f_n(x)\,dx$

(3) Theorem 3 : $\int k f(x)\,dx = k\int f(x)\,dx$ (where k is const.)

Proof :

$$\frac{d}{dx} k\int f(x)\,dx = k\frac{d}{dx}\int f(x)\,dx$$
$$= k f(x) \quad \text{(Rule of derivatives)}$$
$$\therefore \int k f(x)\,dx = k\int f(x)\,dx \quad \text{(By definition of primitive)}$$

(4) Theorem 4 : $g : [\alpha, \beta] \to R$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). $g'(t)$ is continuous on (α, β) and $g'(t) \neq 0$, $\forall\, t \in (\alpha, \beta)$. Range of g is a subset of $[a, b]$ and $f : [a, b] \to R$ is continuous. Then substitution $x = g(t)$ gives,

$$\int f(x)\,dx = \int f[g(t)]\,g'(t)\,dt$$

Proof : Since f is continuous on $[a, b]$, $\int f(x)\,dx$ exists. Since $x = g(t)$ is continuous and $R_g \subset [a, b]$, $f(g(t))$ is defined on $[\alpha, \beta]$ and is continuous on $[\alpha, \beta]$. $g'(t)$ is also continuous on (α, β) and $\int f(g(t))\,g'(t)\,dt$ exists.

Let $h(x) = \int f(x)\,dx$. $\quad \therefore h'(x) = f(x)$.

Hence h is a differentiable function of x on (a,b) and x is a differentiable function of t on (α, β). Hence, h is also a differentiable function of t on (α, β).

$$\therefore \frac{d}{dt} h[g(t)] = h'[g(t)]\,g'(t) = f[g(t)]\,g'(t)$$
$$\therefore h[g(t)] = \int f[g(t)]\,g'(t)\,dt$$
$$\therefore h(x) = \int f[g(t)]\,g'(t)\,dt$$
$$\therefore \int f(x)\,dx = \int f[g(t)]\,g'(t)\,dt$$

As $g'(t)$ is continuous and non-zero, $x = g(t)$ is one-one and hence by $t = g^{-1}(x)$, the function on right-hand side can be converted into a function of x.

5 - 6. INDEFINITE INTEGRATION

(5) Theorem 5 : If $\int f(x)\,dx = F(x)$, then $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$ where $f : I \to R$ is continuous on some interval I. $(a \neq 0)$

Proof : Let $ax+b = t$ $\therefore x = \frac{1}{a}(t-b) = g(t)$ is continuous and differentiable and $g'(t) = \frac{1}{a} \neq 0$. Also $g'(t)$ is continuous.

$$\frac{dx}{dt} = g'(t) = \frac{1}{a}$$

$$\therefore \int f(ax+b)\,dx = \int f(t)\frac{dx}{dt}\,dt = \frac{1}{a}\int f(t)\,dt$$

$$= \frac{1}{a}F(t) = \frac{1}{a}F(ax+b)$$

(6) Theorem 6 : $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$ where $f(x) > 0$ and f, f' are continuous and $f'(x) \neq 0$.

Proof : Let $t = f(x)$. So $\frac{dt}{dx} = f'(x)$

Since $f'(x) \neq 0$ and continuous, $t = f(x)$ is one-one.

$$\therefore \int [f(x)]^n f'(x)\,dx = \int t^n\,dt = \frac{t^{n+1}}{n+1} + c$$

$$= \frac{[f(x)]^{n+1}}{n+1} + c$$

(7) Theorem 7 : If f is continuous and differentiable and f' is continuous and non-zero, $f(x) \neq 0$, $x \in [a, b]$, then $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$.

Proof: f' is continuous and non-zero. Hence f is monotonic. Hence $t = f(x) \Rightarrow x = f^{-1}(t)$. $\therefore f'(x)\frac{dx}{dt} = 1$.

$$\therefore \int \frac{f'(x)}{f(x)}\,dx = \int \frac{f'(x)}{f(x)}\cdot\frac{dx}{dt}\,dt = \int \frac{dt}{t} = \log|t| + c$$

$$= \log|f(x)| + c$$

5 - 6. INDEFINITE INTEGRATION

(8) $\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| = \log|\tan \frac{x}{2}| + c$

Proof: For $\operatorname{cosec} x$ to be defined, $x \neq k\pi$, $k \in Z$.

$\therefore \sin x \neq 0$ and $1 - \cos x \neq 0$.

$\therefore \frac{1-\cos x}{\sin x} = \operatorname{cosec} x - \cot x \neq 0$

Now, $\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$

$$= \int \frac{-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x}{\operatorname{cosec} x - \cot x} dx$$

$$= \int \frac{\frac{d}{dx}(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$$

$$= \log|\operatorname{cosec} x - \cot x| + c$$

Also, $\operatorname{cosec} x - \cot x = \frac{1-\cos x}{\sin x} = \tan \frac{x}{2}$

$\therefore \int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| = \log|\tan \frac{x}{2}| + c.$

(9) $\int \sec x \, dx = \log\left|\tan\left(\pi/4 + x/2\right)\right| + c = \log|\sec x + \tan x| + c$

Proof: For $\sec x$ to be defined, $x \neq (2k+1)\frac{\pi}{2}$, $k \in Z$.

$\therefore \cos x \neq 0$ and $1 + \sin x \neq 0$.

$\therefore \frac{1+\sin x}{\cos x} = \sec x + \tan x \neq 0$

Now, $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\frac{d}{dx}(\sec x + \tan x)}{\sec x + \tan x} dx$$

$$= \log|\sec x + \tan x| + c$$

Also, $\sec x + \tan x = \frac{1+\tan^2 \frac{x}{2}}{1-\tan^2 \frac{x}{2}} + \frac{2\tan \frac{x}{2}}{1-\tan^2 \frac{x}{2}}$

$$= \frac{(1+\tan x/2)^2}{(1-\tan x/2)(1+\tan x/2)} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$$

$$= \tan(\pi/4 + x/2)$$

$$\therefore \int \sec x \, dx = \log|\sec x + \tan x| + c = \log\left|\tan\left(\tfrac{\pi}{4}+\tfrac{x}{2}\right)\right| + c$$

(10) <u>Rule of Integration by Parts:</u>

<u>If (i) f, g are differentiable on interval $I \subset R$ and</u>

<u>(ii) f', g' are continuous on I, then</u>

$$\underline{\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx, \quad x \in I.}$$

<u>Proof:</u> f and g are differentiable functions of x.

$\therefore$ according to working rules of differentiation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Now, f, g, f', g' are given to be continuous on $I \subset R$.

$\therefore fg'$ and $f'g$ are also continuous and hence integrable.

$\therefore$ by definition of integration,

$$f(x)g(x) = \int [f(x)g'(x) + g(x)f'(x)]\, dx$$

$$= \int f(x)g'(x)\, dx + \int g(x)f'(x)\, dx$$

$$\therefore \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \quad \text{... (i)}$$

This is the rule of integration by parts.

Now, let $f(x) = u$ and $g'(x) = v$

$\therefore f'(x) = du/dx$ and $g(x) = \int v\, dx$

Substituting in (i)

$$\int uv\, dx = u\int v\, dx - \int\left[\frac{du}{dx}\int v\, dx\right]dx \quad \text{... (ii)}$$

(11) $\underline{\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + c}$

<u>Proof:</u> Let $I = \int \sqrt{x^2+a^2}\, dx = \int \sqrt{x^2+a^2}\,.\,1\, dx$

$$\therefore I = \sqrt{x^2+a^2}\int 1\, dx - \int\left[\frac{d}{dx}\sqrt{x^2+a^2}\,.\int 1\, dx\right]dx$$

$$I = x\sqrt{x^2+a^2} - \int \frac{1}{2\sqrt{x^2+a^2}} \cdot 2x \cdot x\, dx$$

$$= x\sqrt{x^2+a^2} - \int \frac{x^2\, dx}{\sqrt{x^2+a^2}}$$

$$= x\sqrt{x^2+a^2} - \int \frac{(x^2+a^2) - a^2}{\sqrt{x^2+a^2}}\, dx$$

$$= x\sqrt{x^2+a^2} - \int \sqrt{x^2+a^2}\, dx + a^2 \int \frac{dx}{\sqrt{x^2+a^2}}$$

$$= x\sqrt{x^2+a^2} - I + a^2 \log |x + \sqrt{x^2+a^2}| + 2c$$

$$\therefore \quad 2I = x\sqrt{x^2+a^2} + a^2 \log |x + \sqrt{x^2+a^2}| + 2c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2+a^2}| + c$$

Similarly, it can be proved that

$$\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2-a^2}| + c$$

(12) $\underline{\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c \quad (a > 0)}$

<u>Proof</u>: $I = \int \sqrt{a^2-x^2}\, dx = \int 1 . \sqrt{a^2-x^2}\, dx$

$$= \sqrt{a^2-x^2} \int 1\, dx - \int \left[\frac{d}{dx} \sqrt{a^2-x^2} \int 1\, dx \right] dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{1}{2\sqrt{a^2-x^2}} \cdot (-2x) . x\, dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{(a^2-x^2) - a^2}{\sqrt{a^2-x^2}}\, dx$$

$$= x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\, dx + a^2 \int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= x\sqrt{a^2-x^2} - I + a^2 \sin^{-1}\frac{x}{a} + 2c$$

$$\therefore \quad 2I = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\frac{x}{a} + 2c$$

$$\therefore \quad I = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a} + c$$

(Note: This result can also be proved by the substitution $x = a \sin\theta$.)

5 - 6. INDEFINITE INTEGRATION

(13) $\int e^{x}\,[f(x) + f'(x)]\,dx = e^{x} f(x) + c$

Proof: $\int e^{x} f(x)\,dx = f(x) \int e^{x} dx - \int [\frac{d}{dx} f(x) \int e^{x} dx] dx$

$= e^{x} f(x) - \int e^{x} f'(x)\,dx + c$

$\therefore \int e^{x} [f(x) + f'(x)]\,dx = e^{x} f(x) + c.$

(14) $\int e^{ax} \sin bx\,dx = \dfrac{e^{ax}}{a^2 + b^2} (a \sin bx - b\cos bx)$

$= \dfrac{e^{ax} \sin (bx - \theta)}{\sqrt{a^2+b^2}} + c$

where, $\cos \theta = \dfrac{a}{\sqrt{a^2+b^2}}$, $\sin \theta = \dfrac{b}{\sqrt{a^2+b^2}}$, $a, b \neq 0.$

Proof: $I = \int e^{ax} \sin bx\,dx$

$= \sin bx \int e^{ax} dx - \int [\frac{d}{dx} \sin bx \int e^{ax} dx] dx$

$= \sin bx \cdot \dfrac{e^{ax}}{a} - b \int \cos bx \cdot \dfrac{e^{ax}}{a} dx$

$= \sin bx \cdot \dfrac{e^{ax}}{a} - \dfrac{b}{a} \int e^{ax} \cos bx\,dx$

$\therefore I = \sin bx \cdot \dfrac{e^{ax}}{a} - \dfrac{b}{a} \left[\cos bx \int e^{ax} - \int -b \sin bx \cdot \dfrac{e^{ax}}{a} dx \right]$

$= \sin bx \cdot \dfrac{e^{ax}}{a} - \dfrac{b}{a^2} \cos bx \cdot e^{ax} - \dfrac{b^2}{a^2} \int e^{ax} \sin bx\,dx$

$= \dfrac{e^{ax}}{a^2} (a \sin bx - b \cos bx) - \dfrac{b^2}{a^2} I + \left(\dfrac{a^2+b^2}{a^2} \right) c$

$\therefore \dfrac{a^2+b^2}{a^2} I = \dfrac{e^{ax}}{a^2} (a \sin bx - b \cos bx) + \left(\dfrac{a^2+b^2}{a^2} \right) c$

$\therefore I = \dfrac{e^{ax}}{a^2+b^2} (a \sin bx - b \cos bx) \quad \ldots \text{(i)}$

Now, let $a = r\cos\theta$, $b = r\sin\theta$ $\therefore r = \sqrt{a^2+b^2}$

$\therefore a \sin bx - b\cos bx = r \cos\theta \sin bx - r \sin\theta \cos bx$

$= r \sin (bx - \theta)$

$= \sqrt{a^2+b^2} \sin (bx - \theta)$

Substituting in (i),

$$I = \frac{e^{ax}}{a^2+b^2}\left[\sqrt{a^2+b^2}\,\sin(bx-\theta)\right] + c$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin(bx-\theta) + c$$

where, $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$; $a, b \neq 0$.

Similarly, it can be proved that

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$$

$$= \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos(bx-\theta) + c$$

where, $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$; $a, b \neq 0$.

7 - 9. DEFINITE INTEGRALS
DIFFERENTIAL EQUATIONS

(1) Fundamental principle of definite integration :

If the function f is continuous on $[a, b]$ and F is a differentiable function on $[a, b]$ such that

$F'(x) = f(x), \ \forall \ x \in [a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(2) Rule of substitution in definite integration :

$\phi : [\alpha, \beta] \to [a, b]$ is an increasing (or decreasing) function. $x = \phi(t)$ and $\phi'(t)$, $t \in [\alpha, \beta]$ are continuous and $\phi'(t)$ has constant sign. If $a = \phi(\alpha)$, $b = \phi(\beta)$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f[\phi(t)]\,\phi'(t)\,dt$$

(3) Theorem 1 : If f is an even continuous function on $[-a, a]$

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad (a \in R^+)$$

Proof : $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \quad [\because -a < 0 < a]$

Now, let $I = \int_{-a}^{0} f(x)\,dx$.

Let $x = -t \quad \therefore dx = -dt$

Also $x = -a \Rightarrow t = a$ and $x = 0 \Rightarrow t = 0$

$x = -t$ and $dx/dt = -1$ are continuous on $[-a, a]$ and sign of dx/dt is constant.

$\therefore I = \int_a^0 f(-t)(-dt) = -\int_a^0 f(-t)\,dt = \int_0^a f(t)\,dt$ $(\because f$ is even$)$

$\therefore I = \int_0^a f(x)\,dx$

$\therefore \int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx$

(4) Theorem 2 : If f is continuous and odd function on $[-a,a]$

$$\int_{-a}^{a} f(x)\,dx = 0 \quad (a \in R^+)$$

Proof : $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad [\because -a<0<a]$

Now, let $I = \int_{-a}^{0} f(x)\,dx$

Let $x=-t \qquad \therefore\ dx = -dt$

Also $x=-a \Rightarrow t=a$ and $x=0 \Rightarrow t=0$

$$\therefore I = \int_{a}^{0} f(-t)(-dt) = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt$$

$$= -\int_{0}^{a} f(t)\,dt \quad (\because f \text{ is an odd function})$$

$$= -\int_{0}^{a} f(x)\,dx$$

$$\therefore \int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

(5) Theorem 3 : If f is continuous on $[0,a]$,

$$\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$$

Proof : Let $I = \int_{0}^{a} f(a-x)\,dx$

Let $a-x=t \qquad \therefore\ dx = -dt$

Also, $x=0 \Rightarrow t=a$ and $x=a \Rightarrow t=0$

$$\therefore I = \int_{a}^{0} f(t)(-dt) = -\int_{a}^{0} f(t)\,dt = \int_{0}^{a} f(t)\,dt$$

$$= \int_{0}^{a} f(x)\,dx$$

$$\therefore \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$$

(6) Theorem 4 : If f is continuous over $[a,b]$,

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$$

Proof : Let $I = \int_{a}^{b} f(a+b-x)\,dx$

Let $a+b-x = t \qquad \therefore\ -dx = dt$

Also, $x = a \Rightarrow t = b$ and $x = b \Rightarrow t = a$.

$$\therefore I = \int_b^a f(t)(-dt) = -\int_b^a f(t)\,dt = \int_a^b f(t)\,dt$$

$$= \int_a^b f(x)\,dx$$

$$\therefore \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

(7) Theorem 5 : If f is continuous on $[0, 2a]$,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Proof : $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

$(\because 0 < a < 2a)$

Let $x = 2a - t$ in second integral on R.H.S.

$\therefore dx = -dt$. Also, $x = a \Rightarrow t = a$ and $x = 2a \Rightarrow t = 0$

$$\therefore \int_a^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt) = -\int_a^0 f(2a-t)\,dt$$

$$= \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Corollary : (i) If $f(2a-x) = f(x)$, $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$

(ii) If $f(2a-x) = -f(x)$, $\int_0^{2a} f(x)\,dx = 0$

(8) Differential equation :

A functional relation between independent variable x, dependent variable y and its derivatives $y_1, y_2 \cdots y_n$ is called a differential equation. It is denoted by $F(x, y, y_1, y_2 \cdots y_n) = 0$.

If a differential equation is written in the form of a polynomial, the order of the highest order derivative occuring in the equation is called the order of the differential equation and its power is called the degree of the differential equation.

7 - 9. DEFINITE INTEGRALS DIFFERENTIAL EQUATIONS

Given a differential equation, if we can find a function $y = f(x)$ such that x, y and its derivatives identically satisfy the differential equation, then the function $y = f(x)$ is called the solution of the given differential equation. If the solution contains the constants of integration, it is called the general solution.
The general solution of a differential equation will contain as many constants as the order of the differential equation.

(9) <u>Homogeneous Differential Equations</u> :

If $f(x, y)$ is a function of two variables x and y and it can be written as $x^n \phi(y/x)$ form where $n \in Q$ and ϕ is a function of one variable $t = y/x$, then f is a homogeneous function in x, y having degree n.
If in the differential equation $Mdx + Ndy = 0$, M and N are homogeneous functions of x, y having degree n, it is said to be a homogeneous differential equation.
In other words, if in first order first degree differential equation, $Mdx + Ndy = 0$, M and N are homogeneous functions of the same degree, that is, the equation can be written as

$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, it is a homogeneous differential equation.

(10) <u>Solution of homogeneous differential equation</u>:

Let $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ be a homogeneous differential equation.

To solve such an equation, let $y = vx$

$\therefore \ \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Substituting in differential equation,

$v + x\dfrac{dv}{dx} = f(v) \quad \therefore \ x\dfrac{dv}{dx} = f(v) - v$

$\therefore \ \dfrac{dv}{f(v)-v} = \dfrac{dx}{x}$ is of variables separable type.

Integrating $\displaystyle\int \frac{dv}{f(v)-v} = \log|x| + c$ (c is arbitrary const.)

$\therefore \ F(v) = \log|x| + c$ where, $F(v) = \displaystyle\int \frac{dv}{f(v)-v}$

Substituting $v = y/x$, we get the solution of the differential equation.

(11) Linear Differential Equation and its solution :

The differential equation of one degree in y and its derivatives is called a linear differential equation. Let $\dfrac{dy}{dx} + Py = Q$ be a given linear differential equation of first order, where P, Q are functions of x. We shall consider a special type only where P is a constant.

Multiplying both sides of the equation by $e^{\int P dx} = e^{Px}$,

$$e^{Px}\frac{dy}{dx} + Pe^{Px}y = Qe^{Px}$$

Now, $e^{Px}\dfrac{dy}{dx} + Pe^{Px}y = \dfrac{d}{dx}(e^{Px}y)$

$\therefore \ \dfrac{d}{dx}(e^{Px}y) = Qe^{Px}$, where Qe^{Px} is a function of x only.

Integrating, $e^{Px}y = \displaystyle\int Qe^{Px}dx + c$ (c is arbitrary constant).

This is the general solution.

10 - 11. PROBABILITY

10 - 11. PROBABILITY

Definitions :

1) Random experiment :

An experiment for which all of its outcomes are known in advance but which of these outcomes will occur can be determined only after its performance is called a random experiment.

2) Sample space :

The set of all possible outcomes of a random experiment is called a sample space. A sample space is, usually, denoted by the symbol U and its element is denoted by the letter x.

3) Event :

A subset of the sample space U is called an event. It is usually denoted by capital letters $A, B, C, D, \ldots$ or subscripted symbols $A_1, A_2, A_3 \ldots$

4) Impossible and certain events :

The null set ϕ as typical subset of sample space is called an impossible event. Sample space U is called a certain event.

5) Complementary event :

Let A be an event. Set of all points or elements of U which are not in A is called the complementary event of A. It is denoted by A' or $\bar{A}$. In set theoretic notation, $A' = \{x \mid x \in U, x \notin A\}$. Impossible event ϕ and certain event U are complementary events.

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6) <u>Union of two events</u>:
Let A and B be events. Set of all elements which are either in A or in B is called the union of two events A and B and is denoted by $A \cup B$. In set theoretic notation, $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$

7) <u>Intersection of two events</u>:
Let A and B be events. Set consisting of all elements which are in A and in B is called the intersection of two events A and B and is denoted by $A \cap B$. In set theoretic notation, $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$.

8) <u>Mutually exclusive events</u>:
Let A and B be events. If $A \cap B = \phi$, then A and B are called mutually exclusive events.

9) <u>Exhaustive events</u>:
Let A and B be events. If $A \cup B = U$, then A and B are called exhaustive events.

10) <u>Mutually exclusive and exhaustive events</u>:
Let A and B be events. If $A \cap B = \phi$ and $A \cup B = U$, then A and B are called mutually exclusive and exhaustive events. Obviously, $B = A' = U - A$.

11) <u>Difference of two events</u>:
Let A and B be events. The set of elements which are in A but not in B is called the difference of two events A and B and is denoted by $A - B$. Similarly, we can define the event $B - A$. In set theoretic notation, $A - B = \{x \in U \mid x \in A \text{ and } x \notin B\}$

$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$.

12) Partition of a sample space :
Mutually exclusive and exhaustive events, $A_1, A_2, \dots A_n$ $(n \geq 2)$ form a partition of a sample space U.

13) Elementary events :
Let U be a finite sample space and let $U = \{x_1, x_2, \dots x_m\}$. singleton subsets $\{x_i\}$ of U for $i = 1, 2, \dots, m$ are called elementary events.

14) Set Function :
Let R be the set of real numbers and S be the powerset of U. A rule $T : S \rightarrow R$ which assigns a unique element of R to every member of S is called a set function.

15) Additive set function :
Let $T : S \rightarrow R$ be a set function. If for all $A_1 \in S$, $A_2 \in S$ and $A_1 \cap A_2 = \phi$, $T(A_1 \cup A_2) = T(A_1) + T(A_2)$, then T defined on S is called an additive set function.

16) Axiomatic Definition of Probability :
Let U be a finite sample space and S be the power set of U. Let the set function $P : S \rightarrow R$ satisfy the following axioms :

Axiom 1 : For every $A \in S$, $P(A) \geq 0$.

Axiom 2 : $P(U) = 1$.

Axiom 3 : If $A_1 \in S$ and $A_2 \in S$ are any mutually exclusive events then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.

Set function P defined on S and satisfying the axioms 1, 2 and 3 is called a probability function and for $A \in S$, $P(A)$ is called the probability of event A. The triplet (U, S, P) is called a probability space.

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17) Equally likely elementary events:
Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite sample space. If $P(\{x_1\}) = P(\{x_2\}) = \ldots = P(\{x_n\})$, then elementary events $\{x_1\}, \{x_2\}, \ldots \{x_n\}$ are called equally likely.

18) Classical definition of probability:
If a sample space associated with a random experiment has n possible elements (or n outcomes) and if r $(0 \leq r \leq n)$ out of n elements are favourable to the occurrence of the event A, then $P(A) = \frac{r}{n}$.

19) Conditional probability:
Let A and B be events of the power set S of a finite sample space U and let P be the probability function defined over S. If $P(B) > 0$, then the conditional probability $P(A|B)$ of event A given the event B is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If $P(B) = 0$, $P(A|B)$ cannot be defined.

20) Independence of two events:
Let A and B be events with $P(A) > 0$ and $P(B) > 0$. If the conditional probability $P(A|B)$ of event A given the event B is equal to $P(A)$, i.e., $P(A|B) = P(A)$, then events A and B are said to be independent. Similarly, $P(B|A) = P(B)$ also implies the same condition. Alternately, if $P(A \cap B) = P(A).P(B)$, then events A and B are said to be independent events.

21) Independence of More than Two Events:
For more than two events, mathematician Bernstien

put forward the concept of two types of independence, namely, pairwise independence and mutual independence.

Pairwise independence :

For three events A_1, A_2 and A_3 with $P(A_i) > 0$ $(i = 1, 2, 3)$,

if $P(A_1 \cap A_2) = P(A_1)P(A_2)$

$P(A_2 \cap A_3) = P(A_2)P(A_3)$

$P(A_3 \cap A_1) = P(A_3)P(A_1)$

then events A_1, A_2 and A_3 are said to be pairwise independent.

Mutual independence :

For three events A_1, A_2, A_3 with $P(A_i) > 0$ $(i = 1, 2, 3)$,

if $P(A_1 \cap A_2) = P(A_1)P(A_2)$

$P(A_2 \cap A_3) = P(A_2)P(A_3)$

$P(A_3 \cap A_1) = P(A_3)P(A_1)$ and

$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

then events A_1, A_2, A_3 are said to be mutually independent.

22) Random variable :

Let U be a sample space associated with a random experiment. A real valued function defined on U, $X : U \to R$ is called a random variable.

23) Probability distribution of random variable :

Let $X : U \to R$ be a random variable. Suppose X assumes values in the set $\{x_1, x_2, \ldots, x_n\}$ which is a subset of R. Further suppose that x assumes a value x_i with probability $p(x_i) = P(X = x_i)$. If $p(x_i) \geq 0$, $i = 1, 2 \ldots n$ and $\sum_{i=1}^{n} p(x_i) = 1$, then the set of real values $\{p(x_1), p(x_2), \ldots, p(x_n)\}$ is called a probability distribution of the random variable $\{x_1, x_2, \ldots, x_n\}$

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Theory :

1) Theorem 1 : For impossible event ϕ, $P(\phi) = 0$.

Proof : Events ϕ and U are mutually exclusive and $\phi \cup U = U$. Suppose $A_1 = \phi$ and $A_2 = U$.

Then $P(\phi \cup U) = P(U) = P(\phi) + P(U)$ ($\because$ axiom 3)

$\therefore 1 = P(\phi) + 1$. Hence $P(\phi) = 0$.

2) Theorem 2 : If A is any event, $P(A') = 1 - P(A)$.

Proof : Events A and A' are mutually exclusive and $A \cup A' = U$. Hence by axiom 3, we have

$P(A \cup A') = P(A) + P(A')$

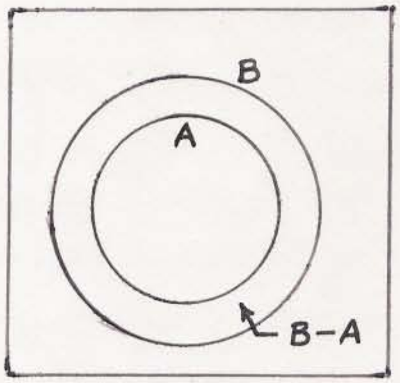
$P(A \cup A') = P(U) = 1$

$\therefore P(A) + P(A') = 1$

$\therefore P(A') = 1 - P(A)$. Also, $P(A) = 1 - P(A')$

3) Theorem 3 : If A and B are events and $A \subset B$, then $P(B-A) = P(B) - P(A)$ and $P(A) \leq P(B)$

Proof : It is given that $A \subset B$. This is shown in the Venn diagram given below.



Now A and B−A are mutually exclusive events and $A \cup (B-A) = B$

$\therefore P(B) = P(A) + P(B-A)$

$\therefore P(B-A) = P(B) - P(A)$

For event B−A, $P(B-A) \geq 0$ and hence $P(A) \leq P(B)$

Thus for $A \subset B$, $P(A) \leq P(B)$.

corollary 1: For event A, $0 \leq P(A) \leq 1$.

Proof : Clearly $\phi \subset A$ and $A \subset U$.

According to axiom $P(A) \geq 0$.

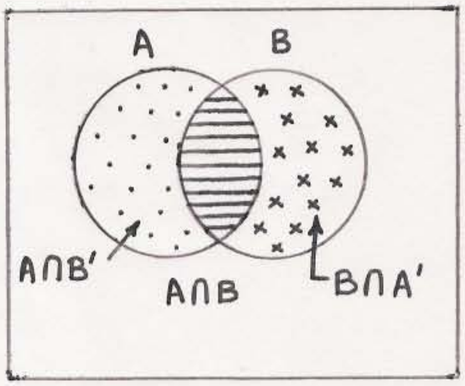
$\therefore\ 0 = P(\phi) \leq P(A)$ and $P(A) \leq P(U) = 1$.

Hence, $0 \leq P(A) \leq 1$.

Corollary 2 : For any events A and B

$$P(A \cap B') = P(A) - P(A \cap B)$$

Proof : Venn diagram for events A and B which are not mutually exclusive is drawn. From the Venn diagram,



$A \cap B' = A - (A \cap B)$

(This result is true even if A and B are mutually exclusive.)

Since $(A \cap B) \subset A$, we have

$$P(A \cap B') = P(A) - P(A \cap B)$$

4) Theorem 4 : If A and B are any events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof : From the above Venn diagram it is clear that events A and B − A are mutually exclusive and $A \cup (B \cap A') = A \cup B$

$\therefore\ P(A \cup B) = P(A) + P(B \cap A')$ ... ...(i)

Since $B \cap A' = B - (A \cap B)$ and $(A \cap B) \subset B$, according to theorem 3, we have

$P(B \cap A') = P(B) - P(A \cap B)$ ... ...(ii)

Hence, from (i) and (ii), we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

5) Theorem 5 : If A, B and C are events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

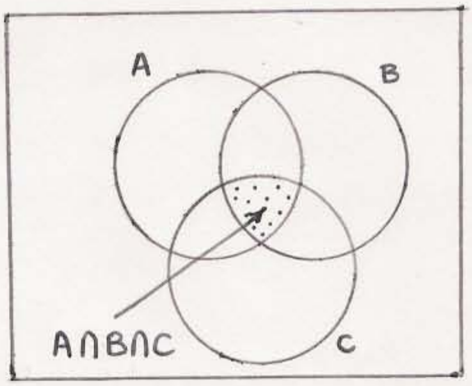
<u>Proof</u> : $P(A \cup B \cup C) = P(A \cup (B \cup C))$

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$= P(A) + P(B) + P(C) - P(B \cap C)$

$- P((A \cap B) \cup (A \cap C))$

($\because$ Theorem 4 and distributive law of set operations).



$$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

(Note : Theorems 4 and 5 are known as addition theorems of probability).

6) <u>To prove that the set function $P(A|B)$ treated as a function of event A for fixed event B is a probability function on S.</u>

<u>Proof</u> : (i) $P(A \cap B) \geq 0$ and $P(B) > 0$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$$

Hence, for a fixed event B in S and for any event $A \in S$, $P(A|B) \geq 0$ which satisfies axiom 1 of probability function.

(ii) If $A = U$, then by definition of $P(A|B)$,

$$P(U|B) = \frac{P(U \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

This satisfies axiom 2 of probability function.

(iii) If A_1 and A_2 are mutually exclusive events in S, then by definition of conditional probability,

$$P(A_1 \cup A_2 | B) = \frac{P[(A_1 \cup A_2) \cap B]}{P(B)}$$

By distributive law of set operations,

$$(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$$

Since events A_1 and A_2 are mutually exclusive, events $A_1 \cap B$ and $A_2 \cap B$ are also mutually exclusive. Hence, by axiom 3 of probability,

$$P[(A_1 \cup A_2) \cap B] = P(A_1 \cap B) + P(A_2 \cap B)$$

$$\therefore P(A_1 \cup A_2 | B) = \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$

$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

Thus, for mutually exclusive events A_1 and A_2,

$$P(A_1 \cup A_2 | B) = P(A_1 | B) + P(A_2 | B).$$

Hence, conditional probability satisfies the axiom 3.

7) $P(A|B) = P(A \cap B)/P(B) \Rightarrow P(A \cap B) = P(B)\,P(A|B)$

Similarly, $P(A \cap B \cap C) = P(A) . P(B|A) . P(C|A \cap B)$

This is called the <u>multiplication rule of probability</u>.

8) <u>Theorem 7 : Bayes' Rule : Let B_1 and B_2 be mutually exclusive and exhaustive events. If $P(B_i)$ and $P(A|B_i)$ for $i = 1, 2$ are given and $P(A) \neq 0$, then</u>

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)}, \quad i = 1, 2$$

<u>Proof</u> : By definition of ~~classical~~ conditional probability,

$$P(B_i|A) = P(A \cap B_i)/P(A) \quad \dots \quad \dots \quad (i)$$

Now, by multiplication rule of probability and theorem 6,

$$P(A \cap B_i) = P(B_i)\,P(A|B_i) \quad \dots \quad \dots \quad \dots (ii)$$

$$P(A) = P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) \quad \dots (iii)$$

Using results (ii) and (iii) in (i), we get

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2)} \quad i = 1, 2$$

For three mutually exclusive and exhaustive events B_1, B_2, B_3,

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + P(B_3)\,P(A|B_3)}$$

$i = 1, 2, 3.$

9) <u>Theorem 6.</u> Let B_1 and B_2 be mutually exclusive and exhaustive events and $P(B_1) \neq 0$ and $P(B_2) \neq 0$. If an event A occurs under the influence of any one of the two events B_1 and B_2, then

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$$

<u>Proof:</u> According to the conditions of the theorem,

$$U = B_1 \cup B_2 \text{ and hence } A = A \cap U = A \cap (B_1 \cup B_2)$$
$$= (A \cap B_1) \cup (A \cap B_2).$$

Since B_1 and B_2 are mutually exclusive events, $A \cap B_1$ and $A \cap B_2$ are also mutually exclusive events.

Hence $P(A) = P(A \cap B_1) + P(A \cap B_2)$ ... ... (i)

According to multiplication rule of probability,

$$P(A \cap B_1) = P(B_1)P(A|B_1), \text{ and}$$
$$P(A \cap B_2) = P(B_2)P(A|B_2) \quad \text{... ... (ii)}$$

Using relation (ii) in (i), we get

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2)$$

This result can be extended for three mutually exclusive and exhaustive events as follows:

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + P(B_3)P(A|B_3).$$

As B_1 and B_2 are mutually exclusive and exhaustive, taking $B_1 = B$ and $B_2 = B'$, we can write

$$P(A) = P(B)P(A|B) + P(B')P(A|B')$$