9. VECTORS

9 - VECTORS

1) Theorem 1 : For non-null vectors $\bar{x}, \bar{y}$ of R^3,
$\bar{x} = \bar{y} \Leftrightarrow |\bar{x}| = |\bar{y}|$ and directions of $\bar{x}, \bar{y}$ are same.

Proof:

($\Rightarrow$) If $\bar{x} = \bar{y}$, then $|\bar{x}| = |\bar{y}|$.
Also, $\bar{x} = 1 \cdot \bar{y}$ and $1 > 0$, so $\bar{x}, \bar{y}$ have same direction.

($\Leftarrow$) Now, $|\bar{x}| = |\bar{y}|$ and $\bar{x}$ and $\bar{y}$ have same direction is given.
Let $\bar{x} = (x_1, x_2, x_3)$ and $\bar{y} = (y_1, y_2, y_3)$.
$\bar{x}$ and $\bar{y}$ have same direction

$$\Rightarrow \bar{x} = k\bar{y}, \quad \text{where } k > 0$$
$$\therefore |\bar{x}| = |k\bar{y}|$$
$$\therefore |\bar{x}| = |k|\,|\bar{y}|$$
$$\therefore |k| = 1 \quad (\because |\bar{x}| = |\bar{y}|$$
$$\therefore k = \pm 1$$

But $k > 0$. so $k = 1$.
Putting this value of k in $\bar{x} = k\bar{y}$,
we get $\bar{x} = \bar{y}$.

2) The Inequality of Schwarz

For real numbers $x_1, x_2, x_3, y_1, y_2, y_3$,

$$(x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1y_1 + x_2y_2 + x_3y_3)^2$$
$$= (x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2$$

is Lagrange's identity which can be easily checked by algebraic simplification of both sides.
Now, for $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3) \in R^3$

$|\bar{x}|^2 = x_1^2 + x_2^2 + x_3^2$, $\quad |\bar{y}|^2 = y_1^2 + y_2^2 + y_3^2$

$\bar{x}\cdot\bar{y} = x_1y_1 + x_2y_2 + x_3y_3$, and

$|\bar{x}\times\bar{y}|^2 = (x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2$.

Substituting in Lagrange's identity, we get

$$|\bar{x}|^2|\bar{y}|^2 - (\bar{x}\cdot\bar{y})^2 = |\bar{x}\times\bar{y}|^2$$

But $|\bar{x}\times\bar{y}| \geqslant 0$. $\quad \therefore\ |\bar{x}|^2|\bar{y}|^2 - (\bar{x}\cdot\bar{y})^2 \geqslant 0$.

$\therefore\ (\bar{x}\cdot\bar{y})^2 \leq |\bar{x}|^2|\bar{y}|^2 \quad$ or $\quad |\bar{x}\cdot\bar{y}| \leq |\bar{x}||\bar{y}|$

This result is known as Schwarz's inequality which can similarly be proved for $\bar{x}, \bar{y} \in R^2$ also.

3) Triangular Inequality

Theorem 2: For $\bar{x}, \bar{y} \in R^3$, $|\bar{x}+\bar{y}| \leq |\bar{x}| + |\bar{y}|$

Proof:
$$\begin{aligned} |\bar{x}+\bar{y}|^2 &= (\bar{x}+\bar{y})\cdot(\bar{x}+\bar{y}) \\ &= \bar{x}\cdot\bar{x} + \bar{x}\cdot\bar{y} + \bar{y}\cdot\bar{x} + \bar{y}\cdot\bar{y} \\ &= |\bar{x}|^2 + 2\bar{x}\cdot\bar{y} + |\bar{y}|^2 \qquad (\because \bar{x}\cdot\bar{y} = \bar{y}\cdot\bar{x}) \\ &\leq |\bar{x}|^2 + 2|\bar{x}\cdot\bar{y}) + |\bar{y}|^2 \\ &\leq |\bar{x}|^2 + 2|\bar{x}||\bar{y}| + |\bar{y}|^2 \qquad [\because \text{Swartz's inequality}] \\ &= (|\bar{x}| + |\bar{y}|)^2 \end{aligned}$$

$\therefore\ |\bar{x}+\bar{y}| \leq |\bar{x}| + |\bar{y}|$.

4) **Theorem 3:** If, for non-null vectors $\bar{x}, \bar{y} \in R^3$, $(x,\hat{\ }y) = \alpha$, then

(1) $\bar{x}\cdot\bar{y} = |\bar{x}||\bar{y}|\cos\alpha$

(2) $|\bar{x}\times\bar{y}| = |\bar{x}||\bar{y}|\sin\alpha$

(3) $\bar{x} \perp (\bar{x}\times\bar{y})$, $\bar{y} \perp (\bar{x}\times\bar{y})$

Proof: (1) Here $(x,\hat{\ }y) = \alpha \quad (\because 0 \leq \alpha \leq \pi)$

$\therefore\ \cos\alpha = \dfrac{\bar{x}\cdot\bar{y}}{|\bar{x}||\bar{y}|} \quad \therefore\ \bar{x}\cdot\bar{y} = |\bar{x}||\bar{y}|\cos\alpha$

(2) Now, $(\bar{x}\cdot\bar{y})^2 = |\bar{x}|^2|\bar{y}|^2\cos^2\alpha$

and $|\bar{x}\times\bar{y}|^2 = |\bar{x}|^2|\bar{y}|^2 - (\bar{x}\cdot\bar{y})^2$

$= |\bar{x}|^2|\bar{y}|^2(1-\cos^2\alpha)$

$= |\bar{x}|^2|\bar{y}|^2\sin^2\alpha$

Hence, $|\bar{x}\times\bar{y}| = |\bar{x}||\bar{y}|\sin\alpha$

($\because \sin\alpha \geqslant 0$ as $0 \leq \alpha \leq \pi$).

(3) Since $\bar{x}\cdot(\bar{x}\times\bar{y}) = 0$ and $\bar{y}\cdot(\bar{x}\times\bar{y}) = 0$, we have $\bar{x} \perp (\bar{x}\times\bar{y})$ and $\bar{y} \perp (\bar{x}\times\bar{y})$.

5) Theorem 4 : (1) Any vectors of R^2 can be expressed, in a unique way, in the form of linear combination of $(1,0)$, $(0,1)$. (2) Any vectors of R^3 can be expressed, in a unique way, in the form of linear combination of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Proof : (1) Let $\bar{x} = (a,b) \in R^2$

Now, $(a,b) = \alpha(1,0) + \beta(0,1) = (\alpha,\beta)$

$\therefore\ \alpha = a$ and $\beta = b$.

Thus, $(a,b) = a(1,0) + b(0,1)$

If, now $(a,b) = p(1,0) + q(0,1)$, then

$(a,b) = (p,q)$

$\therefore\ p = a$ and $q = b$.

Thus, $(a,b) = a(1,0) + b(0,1)$ is also unique.

$\therefore\ \bar{x} = (a,b) = a\bar{i} + b\bar{j}$ is a unique expression of $\bar{x}$ as linear combination of $\bar{i} = (1,0)$ and $\bar{j} = (0,1)$.

(2) Let $\bar{x} = (a,b,c) \in R^3$.

Now, $(a,b,c) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(0,0,1)$

$= (\alpha,\beta,\gamma)$

$\therefore\ \alpha = a$, $\beta = b$ and $\gamma = c$

Thus, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

If, now $(a,b,c) = p(1,0,0) + q(0,1,0) + r(0,0,1)$,
then $(a,b,c) = (p,q,r)$

$\therefore\ p = a,\ q = b$ and $r = c$.

Thus, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$ is also unique.

$\therefore\ \bar{x} = (a,b,c) = a\bar{i} + b\bar{j} + c\bar{k}$ is a unique expression of $\bar{x}$ as linear combination of $\bar{i} = (1,0,0)$, $\bar{j} = (0,1,0)$ and $\bar{k} = (0,0,1)$.

6) <u>Theorem 5 : The necessary and sufficient condition that non-null vectors $\bar{x} = (x_1, x_2)$, $\bar{y} = (y_1, y_2)$ of R^2 be collinear is that $x_1 y_2 - x_2 y_1 = 0$.</u>

<u>Proof :</u> $(\Rightarrow)$ If $\bar{x}$ and $\bar{y}$ are collinear, then for some $k \neq 0$, $\bar{x} = k\bar{y}$

$\therefore\ (x_1, x_2) = k(y_1, y_2)$

$\therefore\ x_1 = k y_1$ and $x_2 = k y_2$

$\therefore\ x_1 y_2 - x_2 y_1 = 0$.

Thus, the condition is necessary.

$(\Leftarrow)$ Now, if $x_1 y_2 - x_2 y_1 = 0$ and if $y_1 \neq 0$ and $y_2 \neq 0$, then $\frac{x_1}{y_1} = \frac{x_2}{y_2} = k$. $\therefore$ we get $\bar{x} = k\bar{y}$.

Again, if $x_1 y_2 - x_2 y_1 = 0$ and $y_1 = 0$, then as $\bar{y} \neq \theta$, $y_2 \neq 0$, so $x_1 = 0$.

Also, since $\bar{x} \neq \theta$ and so $x_2 \neq 0$.

Hence, by taking $x_2/y_2 = k$, we get $\bar{x} = k\bar{y}$

Thus, the condition $x_1 y_2 - x_2 y_1 = 0$ is sufficient for vectors $\bar{x}$ and $\bar{y}$ to be collinear.

7) Theorem 6 : The necessary and sufficient condition that two non-null vectors, $\bar{x}, \bar{y}$ of R^3 be collinear is that $\bar{x} \times \bar{y} = \theta$.

Proof : If $\bar{x}, \bar{y}$ are collinear, then $\bar{x} = k\bar{y}$; $k \neq 0$.

$\therefore \quad \bar{x} \times \bar{y} = (k\bar{y}) \times \bar{y} = k(\bar{y} \times \bar{y}) = k(\theta) = \theta$.

Thus, $\bar{x} \times \bar{y} = \theta$ is a necessary condition.

Now, let $\bar{x} \times \bar{y} = \theta$ and let $\bar{x} = (x_1, x_2, x_3)$, $\bar{y} = (y_1, y_2, y_3)$,

$\Rightarrow (x_2y_3 - x_3y_2)\bar{i} - (x_1y_3 - x_3y_1)\bar{j} + (x_1y_2 - x_2y_1)\bar{k} = (0, 0, 0)$

$\Rightarrow x_1y_2 - x_2y_1 = 0$; $\quad x_2y_3 - x_3y_2 = 0$; $\quad x_1y_3 - x_3y_1 = 0$.

(a) Now, when $y_1, y_2, y_3 \neq 0$, we have $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = k$

Hence, $\bar{x} = k\bar{y}$, $k \neq 0$.

(b) When one of y_1, y_2, y_3, say $y_1 = 0$, then $x_1 = 0$, $x_2 \neq 0$, $x_3 \neq 0$.

$\therefore \quad \frac{x_2}{y_2} = \frac{x_3}{y_3} = k$ and $\bar{x} = k\bar{y}$, $k \neq 0$.

(c) When only two of y_1, y_2, y_3 are zero, say $y_3 \neq 0$, then $x_1 = x_2 = 0$, $x_3 \neq 0$.

$\therefore \quad \frac{x_3}{y_3} = k$ and $\bar{x} = k\bar{y}$. $k \neq 0$.

Hence, $\bar{x}$ and $\bar{y}$ are collinear vectors.

8) Theorem 7 : A necessary condition for three non-null vectors $\bar{x}, \bar{y}, \bar{z}$ to be coplanar is that $\bar{x} \cdot \bar{y} \times \bar{z} =$

Proof : Here, since $\bar{x}, \bar{y}, \bar{z}$ are coplanar, any one of $\bar{x}, \bar{y}, \bar{z}$ can be expressed as linear combination of the other two.

Suppose that $\bar{z} = \alpha\bar{x} + \beta\bar{y}$

Now, $\bar{x} \cdot (\bar{y} \times \bar{z}) = \bar{x} \cdot [\bar{y} \times (\alpha\bar{x} + \beta\bar{y})]$

$= \alpha\,\bar{x} \cdot [\bar{y} \times \bar{x}] + \beta\,\bar{x} \cdot [\bar{y} \times \bar{y}] = 0$

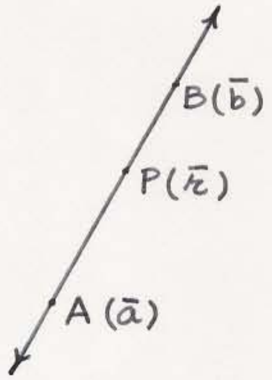
Thus, if $\bar{x}, \bar{y}, \bar{z}$ are coplanar, then $\bar{x} \cdot \bar{y} \times \bar{z} = 0$.

This condition is also sufficient.

10. APPLICATION OF VECTORS

10 - APPLICATION OF VECTORS

(1) Division of a Line Segment



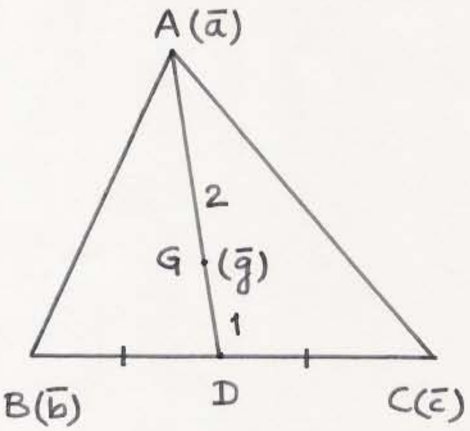
Let the position vectors of A and B be $\bar{a}$ and $\bar{b}$ respectively and let P($\bar{r}$) divide $\overline{AB}$ in the ratio λ from A ($\lambda \neq -1$). Here $\overrightarrow{AP}$ and $\overrightarrow{PB}$ are collinear.

$\therefore \quad \overrightarrow{AP} = \lambda \overrightarrow{PB} \ (\lambda \neq -1) \quad \therefore \ \bar{r} - \bar{a} = \lambda(\bar{b} - \bar{r})$

$$\therefore \quad \bar{r} = \frac{\bar{a} + \lambda \bar{b}}{1+\lambda} = \left[\frac{x_1 + \lambda x_2}{1+\lambda}, \frac{y_1 + \lambda y_2}{1+\lambda}, \frac{z_1 + \lambda z_2}{1+\lambda}\right],$$

where $\bar{a} = (x_1, y_1, z_1)$ and $\bar{b} = (x_2, y_2, z_2)$.

(2) The Centroid of a triangle



Let A($\bar{a}$), B($\bar{b}$) and C($\bar{c}$) be the vertices of ΔABC. Then the position vector of D, the mid-point of $\overline{BC}$ will be $\dfrac{\bar{b}+\bar{c}}{2}$.

Now the centroid G divides $\overline{AD}$ from A in the ratio 2 : 1.

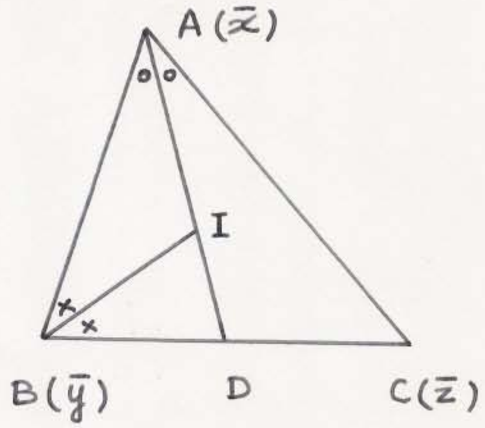
Hence, the position vector of G will be

$$\bar{g} = \frac{1 \cdot \bar{a} + 2\left(\frac{\bar{b}+\bar{c}}{2}\right)}{1+2} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$$

$$= \left[\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right]$$

where $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{c} = (x_3, y_3, z_3)$.

(3) In-centre of a Triangle



Let A ($\bar{x}$), B($\bar{y}$) and C($\bar{z}$) be the vertices of ΔABC and let $AB = c$, $BC = a$ and $CA = b$.

If $\overrightarrow{AD}$ is bisector of $\angle A$, then $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{c}{b}$.

$\therefore$ the position vector of D will be $\dfrac{c\bar{z} + b\bar{y}}{c+b}$.

Now $\overrightarrow{BI}$ is bisector of $\angle B$ in Δ ABD.

$$\therefore \frac{DI}{IA} = \frac{BD}{AB} = \frac{BD}{c}.$$

But $\dfrac{BD}{DC} = \dfrac{c}{b} \Rightarrow \dfrac{BD}{BD + DC} = \dfrac{c}{c+b} \Rightarrow \dfrac{BD}{BC} = \dfrac{c}{c+b}$.

$\therefore BD = \dfrac{ac}{c+b}$. $\quad \therefore \dfrac{DI}{IA} = \dfrac{a}{c+b}$

$\therefore$ I divides $\overline{DA}$ from D in the ratio $a : (c+b)$.

$\therefore$ the position vector of I $= \dfrac{a\bar{x} + (c+b)\left(\frac{c\bar{z}+b\bar{y}}{c+b}\right)}{a+b+c}$

$$= \frac{a\bar{x} + b\bar{y} + c\bar{z}}{a+b+c}.$$

If $\bar{x} = (x_1, y_1, z_1)$, $\bar{y} = (x_2, y_2, z_2)$ and $\bar{z} = (x_3, y_3, z_3)$, then the coordinates of in-centre I of ΔABC are

$$\left[\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right].$$

(4) Area of a Triangle

The area of triangle ABC is

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} |\bar{b}||\bar{c}| \sqrt{1-\cos^2 A} \quad [\because 0 < A < \pi]$$

$$[\bar{b} = \overrightarrow{CA},\ \bar{c} = \overrightarrow{AB},\ \bar{a} = \overrightarrow{BC}].$$

$$= \frac{1}{2} |\bar{b}||\bar{c}| \sqrt{1 - \left[-\frac{\bar{b}.\bar{c}}{|\bar{b}||\bar{c}|}\right]^2}$$

$$= \frac{1}{2}\left[|\bar{b}|^2|\bar{c}|^2 - (\bar{b}.\bar{c})^2\right]^{1/2}$$

Similarly, it can be shown that

$$\Delta = \frac{1}{2}\left[|\bar{c}|^2|\bar{a}|^2 - (\bar{c}.\bar{a})^2\right]^{1/2}$$

$$= \frac{1}{2}\left[|\bar{a}|^2|\bar{b}|^2 - (\bar{a}.\bar{b})^2\right]^{1/2}$$

This formula can be used for triangles in R^2 & R^3.
For triangle in R^3, from Lagrange's identity,

$$|\bar{b} \times \bar{c}|^2 + (\bar{b}.\bar{c})^2 = |\bar{b}|^2|\bar{c}|^2$$

$$\therefore |\bar{b}|^2|\bar{c}|^2 - (\bar{b}.\bar{c})^2 = |\bar{b} \times \bar{c}|^2$$

$$\therefore \Delta = \frac{1}{2}\left[|\bar{b} \times \bar{c}|^2\right]^{1/2} = \frac{1}{2}|\bar{b} \times \bar{c}|$$

$$= \frac{1}{2}|\bar{c} \times \bar{a}| = \frac{1}{2}|\bar{a} \times \bar{b}|$$

This formula for triangle in R^3 can also be used for triangle in R^2 by taking z-coordinates of the vertices as zero.
Here, $\bar{a} = \overrightarrow{BC} = \bar{z} - \bar{y}$; $\bar{b} = \overrightarrow{CA} = \bar{x} - \bar{z}$ and
$\bar{c} = \overrightarrow{AB} = \bar{y} - \bar{x}$ where $\bar{x}, \bar{y}, \bar{z}$ are position vectors of vertices A, B, C of ΔABC.

(5) Sine Formula for a Triangle

Let vertices A, B, C of ΔABC be points of R^3.
The measure of angle between $\bar{b}$ ($\vec{CA}$) and $\bar{c}$ ($\vec{AB}$) in ΔABC is $\pi - A$.

$\therefore |\bar{b} \times \bar{c}| = |\bar{b}||\bar{c}| \sin(\pi - A) = bc \sin A.$

Similarly, $|\bar{c} \times \bar{a}| = ca \sin B$ and $|\bar{a} \times \bar{b}| = ab \sin C$.

Now $\bar{a} + \bar{b} + \bar{c} = \theta \Rightarrow \bar{a} \times (\bar{a} + \bar{b} + \bar{c}) = \bar{a} \times \theta$

$$\Rightarrow \bar{a} \times \bar{b} = -\bar{a} \times \bar{c} = \bar{c} \times \bar{a}$$

Similarly, $\bar{a} \times \bar{b} = \bar{b} \times \bar{c}$

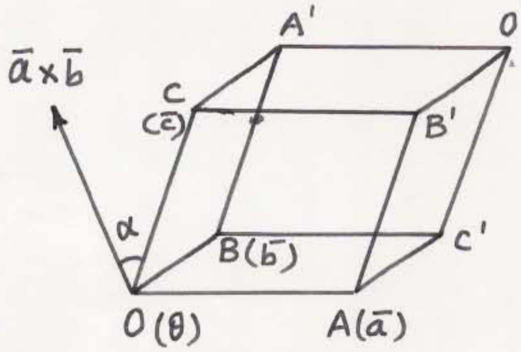
Thus, $\bar{a} \times \bar{b} = \bar{b} \times \bar{c} = \bar{c} \times \bar{a}$

$$\therefore |\bar{a} \times \bar{b}| = |\bar{b} \times \bar{c}| = |\bar{c} \times \bar{a}|$$

$$\therefore ab \sin C = bc \sin A = ca \sin B$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

(6) Volume of a Prism



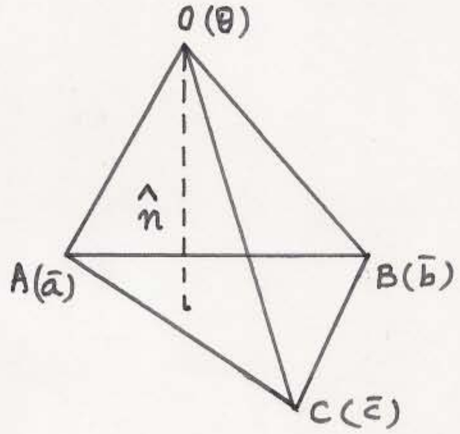
A ($\bar{a}$), B ($\bar{b}$) and C ($\bar{c}$) are vertices of prism OABC-O'A'B'C' with respect to O(θ). For the prism,
area of base $= |\bar{a} \times \bar{b}|$, and
height $= |\bar{c}| \cos \alpha$.

$$\therefore \text{volume, } V = |(\bar{a} \times \bar{b}) \cdot \bar{c}| = \text{modulus of} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

where $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{c} = (x_3, y_3, z_3)$

Thus, if O is one vertex and $\bar{OA}$, $\bar{OB}$, $\bar{OC}$ are edges through O of the prism, then the volume of this prism $= |\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC})|$.

(7) Volume of a Tetrahedron



Let A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) be the vertices of a tetrahedron O-ABC with respect to O($\bar{0}$).

Here, the area of ΔABC

$$= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

If $\hat{n}$ is unit vector perpendicular to the plane of ΔABC, then the height of the tetrahedron $=$ | projection of $\bar{a}$ on $\hat{n}$ | $= |\bar{a} \cdot \hat{n}|$.

Now volume of the tetrahedron

$$= \frac{1}{3} \text{ (height)} \cdot \text{(area of base)}$$

$$= \frac{1}{3} |\bar{a} \cdot \hat{n}| \cdot \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$$

$$= \frac{1}{6} |\bar{a} \cdot (\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})| \quad \left[\because \hat{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|} \right]$$

$$= \frac{1}{6} |\bar{a} \cdot (\bar{b} \times \bar{c} + \bar{c} \times \bar{a} + \bar{a} \times \bar{b})|$$

$$= \frac{1}{6} |\bar{a} \cdot \bar{b} \times \bar{c}| = \frac{1}{6} |[\bar{a} \ \bar{b} \ \bar{c}]|$$

$$= \frac{1}{6} \text{ modulus of } \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \quad \text{where } \bar{a} = (x_1, y_1, z_1), \ \bar{b} = (x_2, y_2, z_2) \text{ and } \bar{c} = (x_3, y_3, z_3).$$

Thus, volume of tetrahedron O-ABC

$$= \frac{1}{6} |\overrightarrow{OA} \cdot \overrightarrow{OB} \times \overrightarrow{OC}|.$$

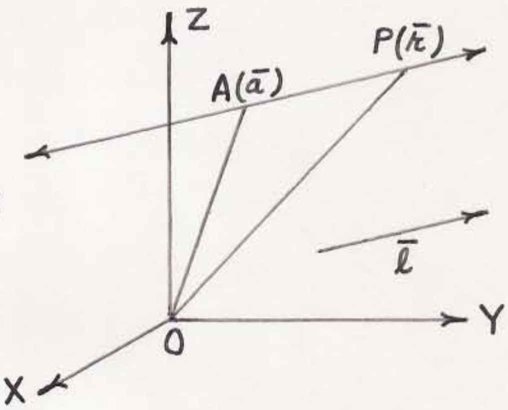
(Note : The derivation can be simplified taking OAB as the base triangle).

11. LINE IN SPACE

(1) Equation of a Line passing through $A(\bar{a})$ and having Direction of the Vector $\bar{l}$

(a) Vector Equation

Here $A(\bar{a})$ is a fixed point of space and $\bar{l}$ is the direction of a non-null vector. Let $P(\bar{r})$ be any point on a line passing through $A(\bar{a})$ and parallel to $\bar{l}$.



Now, P is on a line through $A(\bar{a})$ parallel to $\bar{l}$ and $P \neq A$.

$\Leftrightarrow$ $\overrightarrow{AP}$ and $\bar{l}$ are parallel vectors

$\Leftrightarrow$ $\overrightarrow{AP} = k\bar{l}, \quad k \in R - \{0\}$

$\Leftrightarrow$ $\overrightarrow{OP} - \overrightarrow{OA} = k\bar{l}, \quad k \in R - \{0\}$

$\Leftrightarrow$ $\bar{r} - \bar{a} = k\bar{l}, \quad k \in R - \{0\}$

Also $k = 0 \Rightarrow \bar{r} = \bar{a} \Rightarrow P = A$.

Hence for all points on line $\bar{r} = \bar{a} + k\bar{l}, \; k \in R$.

This is the vector equation of a line passing through $A(\bar{a})$ and parallel to $\bar{l}$.

(b) Cartesian Equations

Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$ and $\bar{l} = (a, b, c)$ in the vector equation $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ of the line through $A(\bar{a})$ having direction of the vector $\bar{l}$, we have

$$(x, y, z) = (x_1, y_1, z_1) + k(a, b, c), \quad k \in R$$

$$= (x_1 + ka, \; y_1 + kb, \; z_1 + kc), \quad k \in R.$$

Hence, the parametric equations of this line are

$$x = x_1 + ka, \quad y = y_1 + kb, \quad z = z_1 + kc, \quad k \in R.$$

Eliminating parameter k from these equations, we get the equations of line through (x_1, y_1, z_1) with direction ratios $a : b : c$ in the form

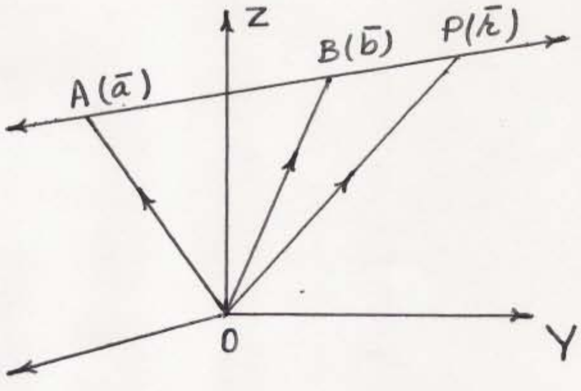
$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}.$$

If l, m, n are direction cosines of the line, then the equations of the line are of form

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{where } l^2+m^2+n^2 = 1$$

(2) Equation of the Line through $A(\bar{a})$, $B(\bar{b})$

(a) Vector Equation



Let A, B be two distinct points in space having position vectors $\bar{a} = (x_1, y_1, z_1)$ and $\bar{b} = (x_2, y_2, z_2)$ respectively.

Let P be any point on the line $\overleftrightarrow{AB}$ having position vector $\bar{r} = (x, y, z)$, and $P \neq A$.

P is a point on line $\overleftrightarrow{AB} \Leftrightarrow \overrightarrow{AP}$ and $\overrightarrow{AB}$ are parallel

$\Leftrightarrow \overrightarrow{AP} = k\overrightarrow{AB}, \; k \in R - \{0\}$

$\Leftrightarrow \bar{r} - \bar{a} = k(\bar{b} - \bar{a}), \; k \in R - \{0\}$

Also if $K = 0$, then $P = A$. Hence for all points on line $\bar{r} = \bar{a} + K(\bar{b} - \bar{a})$, $K \in R$. This is the vector equation of a line through $A(\bar{a})$ and $B(\bar{b})$.

(b) Cartesian Equations

$\bar{r} = \bar{a} + k(\bar{b} - \bar{a})$, $k \in R$ is the vector equation of a line through $A(\bar{a})$ and $B(\bar{b})$.

Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$ and $\bar{b} = (x_2, y_2, z_2)$,

we get $(x, y, z) = (x_1, y_1, z_1) + k(x_2 - x_1, y_2 - y_1, z_2 - z_1)$, $k \in R$

$$\therefore (x - x_1, y - y_1, z - z_1) = k(x_2 - x_1, y_2 - y_1, z_2 - z_1),\ k \in R.$$

$\therefore x - x_1 = k(x_2 - x_1);\ y - y_1 = k(y_2 - y_1);\ z - z_1 = k(z_2 - z_1),\ k \in R.$

Eliminating parameter k from these parametric equations, we get

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

which are cartesian equations of the line through A and B.

(3) Collinear Points

If three distinct points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ of space are collinear, then $C(\bar{c})$ should satisfy the equation of line through $A(\bar{a})$ and $B(\bar{b})$, i.e., $\bar{r} = \bar{a} + k(\bar{b} - \bar{a})$, $k \in R$. Thus, the necessary and sufficient condition is $\bar{c} - \bar{a} = k(\bar{b} - \bar{a})$ for some $k \in R$. This means $\bar{c} - \bar{a}$ and $\bar{b} - \bar{a}$ are parallel, i.e., $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \theta$.

Now, $(\bar{c} - \bar{a}) \times (\bar{b} - \bar{a}) = \theta \Leftrightarrow \bar{c} \times \bar{b} - \bar{a} \times \bar{b} - \bar{c} \times \bar{a} = \theta$

$$\Leftrightarrow \bar{b} \times \bar{c} = \bar{a} \times \bar{c} - \bar{a} \times \bar{b}$$

$$\Rightarrow a \cdot (\bar{b} \times \bar{c}) = \bar{a} \cdot (\bar{a} \times \bar{c}) - \bar{a} \cdot (\bar{a} \times \bar{b}) = 0$$

$$\Rightarrow [\bar{a}\ \bar{b}\ \bar{c}] = 0.$$

As the converse of this may not be true, this condition is only necessary, but not sufficient.

Theorem : Three distinct points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ in space are collinear if and only if non-zero real numbers l, m, n can be so obtained that

$$l\bar{a} + m\bar{b} + n\bar{c} = \theta \text{ and } l+m+n = 0.$$

Proof: Suppose A, B, C are collinear.

$\therefore$ C $(\bar{c})$ satisfies the equation $\bar{r} - \bar{a} = k(\bar{b}-\bar{a})$, $k \in R$ of the line $\overleftrightarrow{AB}$.

$$\therefore \bar{c} - \bar{a} = k(\bar{b}-\bar{a})$$

$$\therefore (k-1)\bar{a} + (-k)\bar{b} + \bar{c} = \theta$$

For $k=1$, $\bar{b} = \bar{c}$ and for $k=0$, $\bar{a} = \bar{c}$. But A, B, C are distinct points. Hence $k \neq 0$, $k \neq 1$.

Taking $l = k-1$, $m = -k$ and $n = 1$,

$$l\bar{a} + m\bar{b} + n\bar{c} = \theta \text{ and } l+m+n = 0.$$

For sufficiency of the conditions, suppose we have real numbers l, m, n so that

$$l\bar{a} + m\bar{b} + n\bar{c} = \theta \text{ and } l+m+n = 0$$

Taking $l = -(m+n)$,

$$-(m+n)\bar{a} + m\bar{b} + n\bar{c} = \theta$$

$$\therefore m(\bar{b}-\bar{a}) + n(\bar{c}-\bar{a}) = \theta$$

$$\therefore \bar{c} - \bar{a} = -\frac{m}{n}(\bar{b}-\bar{a}) = k(\bar{b}-\bar{a}) \text{ where } k = -\frac{m}{n}$$

As $m \neq 0$, $k \neq 0$. Also $k \neq 1$ for if $k=1$, then $n+m=0$ and from $l+m+n=0$, we get $l=0$ which is contradiction, as l, m, n are non-zero numbers. Thus, $k \neq 0$ and $k \neq 1$. Thus $\bar{r} = \bar{c}$ satisfies $\bar{r} = \bar{a} + k(\bar{b}-\bar{a})$, the equation of the line $\overleftrightarrow{AB}$. Thus, C is on the line $\overleftrightarrow{AB}$ and $C \neq A$ or $C \neq B$ since $k \neq 0$, $k \neq 1$.

11. LINE IN SPACE

(4) Angle between Two Lines in R^3

Suppose $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ are given distinct lines.

(a) If $\bar{l} = k\bar{m}$, $k \in R - \{0\}$ or $\bar{l} \times \bar{m} = \theta$, then the two given lines are parallel.

(b) If $\bar{l} \cdot \bar{m} = 0$, the lines are perpendicular.

(c) In all other cases, the acute angle α between the intersecting or skew lines is given by

$$\cos\alpha = \frac{|\bar{l} \cdot \bar{m}|}{|\bar{l}|\,|\bar{m}|}, \quad 0 < \alpha < \frac{\pi}{2}.$$

(5) Condition for Two Lines in R^3 to Intersect

Two given distinct lines in space may be parallel or may not be parallel. They may or may not intersect even when they are not parallel.

Theorem: A necessary condition that two non-parallel lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ in R^3 intersect is that $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$.

Proof: As the lines are not parallel, $\bar{l} \times \bar{m} \neq \theta$. If the lines intersect at $C(\bar{c})$, then

$\bar{c} = \bar{a} + k_1\bar{l}$ and $\bar{c} = \bar{b} + k_2\bar{m}$ for some $k_1, k_2 \in R$.

$\therefore\ \bar{a} - \bar{b} = k_2\bar{m} - k_1\bar{l}$

$\therefore\ (\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = (k_2\bar{m} - k_1\bar{l}) \cdot (\bar{l} \times \bar{m}) = 0$

Thus $(\bar{a} - \bar{b}) \cdot (\bar{l} \times \bar{m}) = 0$ is a necessary condition for two lines to intersect. Taking $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{l} = (l_1, m_1, n_1)$ and $\bar{m} = (l_2, m_2, n_2)$,

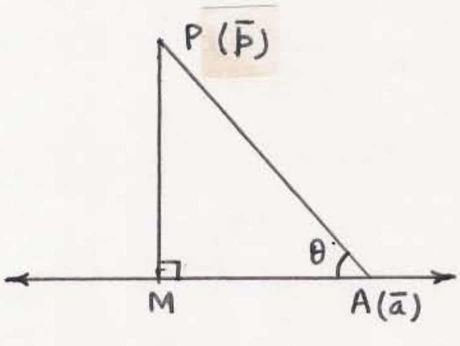
$$(\bar{a}-\bar{b}).(\bar{l}\times\bar{m}) = 0 \Rightarrow \begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

This is the cartesian form of the necessary condition for two lines to intersect.

(6) Coplanar and Non-coplanar (skew) Lines

If the two lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ are parallel or intersecting, then they are coplanar. $\bar{l}\times\bar{m} = \theta$ is the condition for the lines to be parallel and $(\bar{a}-\bar{b}).(\bar{l}\times\bar{m}) = 0$ is the condition for them to be intersecting. Hence, the required condition for the two given lines to be coplanar is $(\bar{a}-\bar{b}).(\bar{l}\times\bar{m}) = 0$. If this condition is not satisfied, then the lines are non-coplanar or skew, i.e., they are neither parallel nor intersecting.

(7) (Perpendicular) Distance of a Point from a Line, in R^3



Suppose $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ is a given line and P $(\bar{p})$ is any point in space not on this line. If it is on the line, then distance is zero.

$\overline{PM}$ is drawn perpendicular to the line.

Now, $|\overrightarrow{AP}\times\bar{l}| = |\overrightarrow{AP}||\bar{l}|\sin\theta = |\bar{l}|\,AP\sin\theta = |\bar{l}|.PM$

$\therefore$ perpendicular distance,

$$PM = \frac{|\overrightarrow{AP}\times\bar{l}|}{|\bar{l}|} = |\overrightarrow{AP}\times\hat{l}|$$

$$= |(\bar{p}-\bar{a})\times\hat{l}|.$$

Taking $\bar{p} = (x_1, y_1, z_1)$, $\bar{a} = (a, b, c)$ and $\hat{l} = (l, m, n)$,

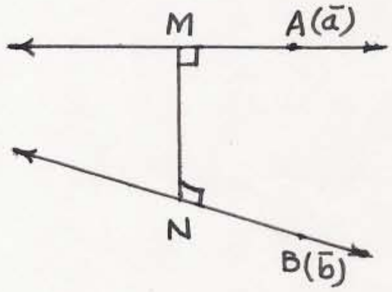
$PM = |\overrightarrow{AP} \times \hat{l}|$ = magnitude of the vector given by

$$\begin{vmatrix} i & j & k \\ x_1 - a & y_1 - b & z_1 - c \\ l & m & n \end{vmatrix}$$

where l, m, n are direction cosines of the line, i.e., $\hat{l} = (l, m, n)$

The distance between two parallel lines $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{l}$, $k \in R$ is $|(\bar{b} - \bar{a}) \times \hat{l}|$.

(8) Shortest Distance between Two Skew Lines



Let $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ be two given skew lines passing through points $A(\bar{a})$ and $B(\bar{b})$ respectively. There are unique points M, N, one on each line such that $\overline{MN}$ is perpendicular to both the given lines and MN is the shortest distance between them.

Unit vector in direction of $\overleftrightarrow{MN}$ is $\pm \dfrac{\bar{l} \times \bar{m}}{|\bar{l} \times \bar{m}|}$.

Now, $MN = |$ projection of $\overrightarrow{AB}$ on $\overleftrightarrow{MN}|$

$$= \left| \overrightarrow{AB} \cdot \frac{\bar{l} \times \bar{m}}{|\bar{l} \times \bar{m}|} \right| = \left| \frac{(\bar{b} - \bar{a}) \cdot (\bar{l} \times \bar{m})}{|\bar{l} \times \bar{m}|} \right|$$

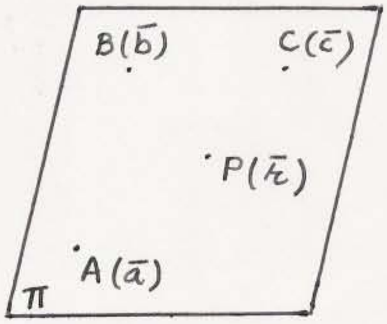
$$= |(\bar{b} - \bar{a}) \cdot \bar{u}| \quad \text{where } \bar{u} = \frac{\bar{l} \times \bar{m}}{|\bar{l} \times \bar{m}|}.$$

12. PLANE

12. PLANE

(1) Equation of a Plane passing through Three Distinct Non-collinear Points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$

(a) Vector Equation



Let π be the unique plane passing through three non-collinear points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$. Let $P(\bar{r})$ be any point of this plane other than A, B, C.

Now, $P \in \pi \Leftrightarrow \overrightarrow{AP}, \overrightarrow{AB}, \overrightarrow{AC}$ are coplanar

$$\Leftrightarrow (\bar{r}-\bar{a}) \cdot [(\bar{b}-\bar{a}) \times (\bar{c}-\bar{a})] = 0$$

$$\Leftrightarrow \bar{r}-\bar{a} = m(\bar{b}-\bar{a}) + n(\bar{c}-\bar{a}), \quad m, n \in R$$

$$\Leftrightarrow \bar{r} = \bar{a} + m(\bar{b}-\bar{a}) + n(\bar{c}-\bar{a}), \quad m, n \in R$$

which is the required equation of the plane. Here, $m=0, n=0$; $m=1, n=0$; $m=0, n=1$ give $\bar{r} = \bar{a}, \bar{b}, \bar{c}$. Hence this plane passes through A, B, C.

(b) Parametric Vector Equation

Vector equation of the plane through $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$

is $\bar{r} = \bar{a} + m(\bar{b}-\bar{a}) + n(\bar{c}-\bar{a})$; $m, n \in R$

$$= (1-m-n)\bar{a} + m\bar{b} + n\bar{c}, \quad m, n \in R$$

$$= l\bar{a} + m\bar{b} + n\bar{c}, \quad l = 1-m-n, \quad m, n \in R.$$

Thus, $\bar{r} = l\bar{a} + m\bar{b} + n\bar{c}$, $l+m+n=1$, $l, m, n \in R$ is parametric vector equation of the plane through A, B, C.

(c) Cartesian Equation

The vector equation of plane through $A(\bar{a})$, $B(\bar{b})$ and

$C(\bar{c})$ is $(\bar{r}-\bar{a}).[(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})]=0.$

Taking $\bar{r}=(x,y,z)$, $\bar{a}=(x_1,y_1,z_1)$, $\bar{b}=(x_2,y_2,z_2)$, and $\bar{c}=(x_3,y_3,z_3)$,

$$(\bar{r}-\bar{a}).\ [(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})]=0$$

$$\Rightarrow [\,\bar{r}-\bar{a},\ \bar{b}-\bar{a},\ \bar{c}-\bar{a}\,]=0$$

$$\Rightarrow \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

This is the cartesian equation of the plane through A, B, C.

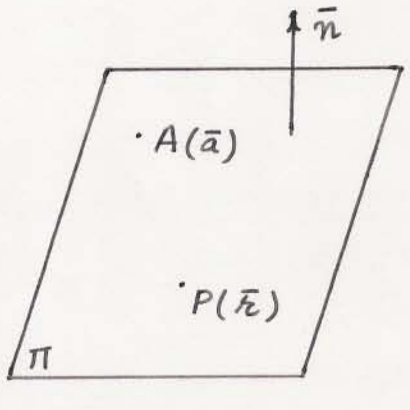
(d) Parametric Cartesian Equations

Parametric vector equation of the plane through $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ is $\bar{r}=l\bar{a}+m\bar{b}+n\bar{c}$, $l+m+n=1$. Taking $\bar{r}=(x,y,z)$, $\bar{a}=(x_1,y_1,z_1)$, $\bar{b}=(x_2,y_2,z_2)$ and $\bar{c}=(x_3,y_3,z_3)$, we get the parametric cartesian form of equations:

$$\left.\begin{aligned} x &= lx_1+mx_2+nx_3 \\ y &= ly_1+my_2+ny_3 \\ z &= lz_1+mz_2+nz_3 \end{aligned}\right\}\ l+m+n=1,\ l,m,n\in R.$$

(2) Equation of the Plane through $A(\bar{a})$ having Normal $\bar{n}$

(a) Vector Equation



Let π be a plane through $A(\bar{a})$ having normal $\bar{n}$. Let $P(\bar{r})$ be any point in the plane other than A.

$$P \in \pi \Leftrightarrow \overrightarrow{AP} \cdot \bar{n} = 0$$
$$\Leftrightarrow (\bar{r} - \bar{a}) \cdot \bar{n} = 0$$
$$\Leftrightarrow \bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$$
$$\Leftrightarrow \bar{r} \cdot \bar{n} = d, \quad \text{where } d = \bar{a} \cdot \bar{n}$$

This is the vector form of the equation of the plane through $A(\bar{a})$ having normal $\bar{n}$.

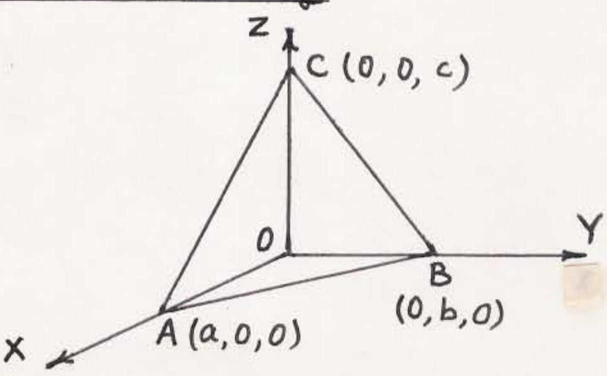
(b) Cartesian Equation

Vector equation of the plane through $A(\bar{a})$ having normal $\bar{n}$ is $(\bar{r} - \bar{a}) \cdot \bar{n} = 0$. Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$ and $\bar{n} = (l, m, n)$, the cartesian equation of this plane is obtained as

$$(x - x_1) l + (y - y_1) m + (z - z_1) n = 0$$
$$\therefore lx + my + nz = lx_1 + my_1 + nz_1$$
$$\therefore lx + my + nz = p \quad \text{where } p = lx_1 + my_1 + nz_1$$

which is the required cartesian equation.
General equation of the plane can be written as $ax + by + cz + d = 0$ or $ax + by + cz + 1 = 0$ where (a, b, c) is normal to the plane. Equation of a plane passing through the origin is of form $ax + by + cz = 0$.

(3) Equation of Plane making intercepts a, b, c on X-, Y-, and Z-axes respectively



The plane passes through $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Hence, its cartesian equation is

$$\begin{vmatrix} x-a & y-0 & z-0 \\ 0-a & b-0 & 0-0 \\ 0-a & 0-0 & c-0 \end{vmatrix} = 0$$

$\therefore \; bc(x-a) - y(-ac) + z(ab) = 0$

$\therefore \; bcx + cay + abz = abc$

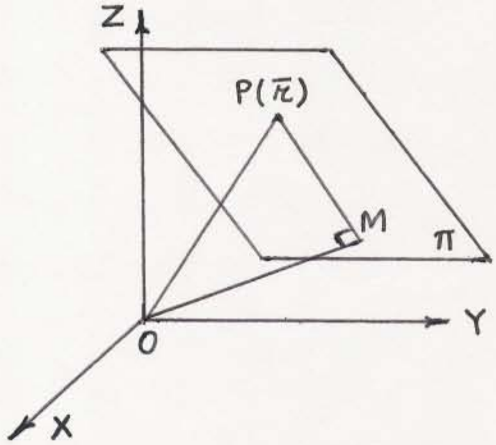
$\therefore \; \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

The parametric cartesian equations of this plane is

$x = la, \quad y = mb, \quad z = nc$ where $l+m+n=1, \; l, m, n \in R$.

Elimination of parameters gives the cartesian equation.

(4) Equation of a Plane Perpendicular on which from Origin O is of Length p and has Direction Angles α, β, γ



Let $P(\bar{r})$ be a point of the plane π on which $OM = p$ is the perpendicular from the origin. The unit vector $\hat{n}$ in the direction of $\overrightarrow{OM}$ is $\hat{n} = (\cos\alpha, \cos\beta, \cos\gamma)$. '$p$' is the projection of $\overrightarrow{OP}$ in direction $\hat{n}$.

$\therefore \; \bar{r} \cdot \hat{n} = p$ is the vector equation of the required plane. Taking $\bar{r} = (x, y, z)$ and $\hat{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ the cartesian equation of the plane is

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p.$$

(5) Equation of a Plane through Two Parallel Lines

Suppose $\bar{r} = \bar{a} + k\bar{l}, \; k \in R$ and $\bar{r} = \bar{b} + k\bar{l}, \; k \in R$

are two parallel lines. The direction of normal to the plane $\bar{n} = (\bar{b}-\bar{a}) \times \bar{l}$. The plane through these two parallel lines passes through $\bar{a}$. Hence its equation is

$$(\bar{r}-\bar{a}).\bar{n} = 0 \quad \text{or,} \quad (\bar{r}-\bar{a}).[(\bar{b}-\bar{a}) \times \bar{l}] = 0.$$

Now, taking $\bar{r}-\bar{a} = k\bar{l}$ in $(\bar{r}-\bar{a}).\bar{n}$, we get

$(\bar{r}-\bar{a}).\bar{n} = (k\bar{l}).[(\bar{b}-\bar{a}) \times \bar{l}] = 0$, and taking $\bar{r} = \bar{b} + k\bar{l}$ in $(\bar{r}-\bar{a}).\bar{n}$, we again get

$$\begin{aligned}(\bar{r}-\bar{a}).\bar{n} &= (\bar{b}-\bar{a}+k\bar{l}).[(\bar{b}-\bar{a}) \times \bar{l}] \\ &= (\bar{b}-\bar{a}).[(\bar{b}-\bar{a}) \times \bar{l}] + k\bar{l}.[(\bar{b}-\bar{a}) \times \bar{l}] \\ &= 0 + 0 = 0.\end{aligned}$$

So every point on each of the two given lines satisfy the equation $(\bar{r}-\bar{a}).\bar{n} = 0$ i.e., $(\bar{r}-\bar{a})[(\bar{b}-\bar{a}) \times \bar{l}] = 0$ which is the required equation of the plane.

Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$ and $\bar{l} = (l, m, n)$, the equation in cartesian form is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$$

(6) Equation of a Plane through Two Intersecting Lines

Let $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ and $\bar{r} = \bar{b} + k\bar{m}$, $k \in R$ be two intersecting lines determining the plane π. Hence, the direction of normal to the plane π is $\bar{n} = \bar{l} \times \bar{m}$. The plane through these two intersecting lines passes through $\bar{a}$. Hence, its equation is $(\bar{r}-\bar{a}).(\bar{l} \times \bar{m}) = 0$.

Now, taking $\bar{r} = \bar{a} + k\bar{l}$ in $(\bar{r}-\bar{a}).(\bar{l} \times \bar{m}) = 0$, we get

$(\bar{r}-\bar{a}).(\bar{l} \times \bar{m}) = k\bar{l}.(\bar{l} \times \bar{m}) = 0$, and taking $\bar{r} = \bar{b} + k\bar{m}$ in $(\bar{r}-\bar{a}).(\bar{l} \times \bar{m}) = 0$, we again get

$$\begin{aligned}(\bar{r}-\bar{a}).(\bar{l} \times \bar{m}) = (\bar{b}-\bar{a}+k\bar{m}).(\bar{l} \times \bar{m}) &= (\bar{b}-\bar{a}).(\bar{l} \times \bar{m}) + k\bar{m}.(\bar{l} \times \bar{m}) \\ &= 0.\end{aligned}$$

Thus every point on each of the given lines satisfies the equation $(\bar{r}-\bar{a})\cdot(\bar{l}\times\bar{m}) = 0$.

Taking $\bar{r} = (x, y, z)$, $\bar{a} = (x_1, y_1, z_1)$, $\bar{l} = (l_1, m_1, n_1)$ and $\bar{m} = (l_2, m_2, n_2)$, the equation in cartesian form is obtained as

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

(7) Angle between the Two Planes $(\bar{r}-\bar{a})\cdot\bar{n}_1 = 0$ and $(\bar{r}-\bar{b})\cdot\bar{n}_2 = 0$

Angle between the two planes is the angle between their normals, $\bar{n}_1$ and $\bar{n}_2$.

(a) If $\bar{n}_1 = k\bar{n}_2$, $k \in R$, or, $\bar{n}_1 \times \bar{n}_2 = \theta$, then the two planes are parallel (or same).

(b) If $\bar{n}_1 \cdot \bar{n}_2 = 0$, then the two planes are mutually perpendicular.

(c) In all other cases, the acute angle α between the two planes is given by

$$\cos\alpha = \frac{|\bar{n}_1 \cdot \bar{n}_2|}{|\bar{n}_1|\,|\bar{n}_2|}, \quad 0 < \alpha < \frac{\pi}{2}$$

(8) Conditions for Two Planes to be Parallel and Perpendicular

Consider the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. Their normals are $\bar{n}_1 = (a_1, b_1, c_1)$ and $\bar{n}_2 = (a_2, b_2, c_2)$ respectively.

(a) If $\bar{n}_1 = k\bar{n}_2$, $k \in R$ or if $\bar{n}_1 \times \bar{n}_2 = \theta$, i.e., if $(a_1, b_1, c_1) = k(a_2, b_2, c_2)$, $k \in R$ or $a_1b_2 - a_2b_1 = 0$, $b_1c_2 - b_2c_1 = 0$ and $c_1a_2 - c_2a_1 = 0$, then the planes are parallel (or same). For planes to be same, an additional condition, $c_1 d_2 - c_2 d_1 = 0$ should be satisfied.

(b) If $\bar{n}_1 \cdot \bar{n}_2 = 0$, i.e., if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$, then the two planes are perpendicular. In all other cases, the acute angle, α, between the planes is given by

$$\cos\alpha = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}, \quad 0 < \alpha < \frac{\pi}{2}.$$

(9) Condition for Four Points in R^3, No Three of which are Collinear, to be Coplanar

Let $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$, $D(\bar{d})$ be coplanar. Hence, the position vector $\bar{d}$ of D will satisfy the equation of the plane through A, B, C. Therefore, writing $\bar{d}$ for $\bar{r}$ in

$(\bar{r} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})] = 0$, we get the necessary and sufficient condition for A, B, C, D to be coplanar as

$(\bar{d} - \bar{a}) \cdot [(\bar{b} - \bar{a}) \times (\bar{c} - \bar{a})] = 0.$

Taking $\bar{a} = (x_1, y_1, z_1)$, $\bar{b} = (x_2, y_2, z_2)$, $\bar{c} = (x_3, y_3, z_3)$ and $\bar{d} = (x_4, y_4, z_4)$, this condition can be shown as:

$$\begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

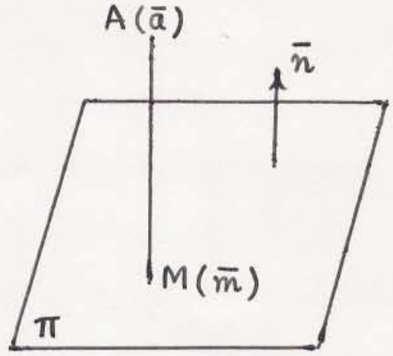
(10) Distance of a given Plane from a given Point in R^3

Let $\bar{r} \cdot \bar{n} = d$ be the equation of a given plane π and $A(\bar{a})$ be a point in R^3. Suppose $A \notin \pi$. Draw $\overline{AM} \perp \pi$.

The equation of $\overleftrightarrow{AM}$ is $\bar{r} = \bar{a} + k\bar{n}$, $k \in R$.

Now, $M(\bar{m})$ is on line $\overleftrightarrow{AM}$ as well as in plane π.

$\therefore\ \bar{m}\cdot\bar{n} = d$ and $\bar{m} = \bar{a} + k_1\bar{n}$ for some $k_1 \in R$.



$\therefore (\bar{a} + k_1\bar{n})\cdot\bar{n} = d$

$$\therefore\ k_1 = \frac{d - \bar{a}\cdot\bar{n}}{|\bar{n}|^2}$$

This value of k_1 gives the foot of perpendicular, $M(\bar{m})$.

Now, $AM = |\bar{m} - \bar{a}| = |k_1\bar{n}| = |k_1||\bar{n}|$

$$= \frac{|d - \bar{a}\cdot\bar{n}|}{|\bar{n}|^2}\cdot|\bar{n}| = \frac{|d - \bar{a}\cdot\bar{n}|}{|\bar{n}|}$$

which is the length of perpendicular from A on plane π.
If the equation of the plane is $ax + by + cz + d = 0$ and if $A(\alpha, \beta, \gamma)$ is the given point, then $\bar{n} = (a, b, c)$, $\bar{a} = (\alpha, \beta, \gamma)$ and the equation of the plane is $\bar{r}\cdot\bar{n} + d = 0$, i.e., $\bar{r}\cdot\bar{n} = -d$.
$\therefore$ perpendicular distance of the point A from the plane

$$= \frac{|\bar{a}\cdot\bar{n} + d|}{|\bar{n}|} = \frac{|(\alpha, \beta, \gamma)\cdot(a, b, c) + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|a\alpha + b\beta + c\gamma + d|}{\sqrt{a^2 + b^2 + c^2}}$$

The perpendicular distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$p = \frac{|d_1 - d_2|}{|\bar{n}|} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

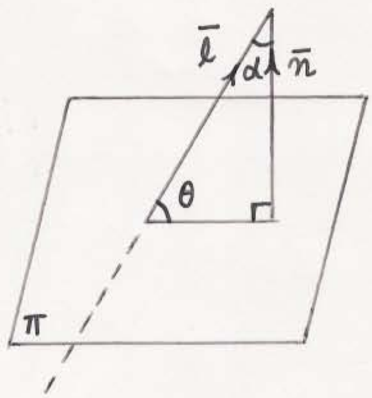
(11) Angle between a Line and a Plane

Let $\bar{r} = \bar{a} + k\bar{l}$, $k \in R$ be the given line and $\bar{r}\cdot\bar{n} = d$ be the given plane.

(a) If $\bar{l}\cdot\bar{n} = 0$, then the line is parallel to the plane and if in addition $\bar{a}\cdot\bar{n} = d$, then the line lies in the plane.

(b) If $\bar{l} = k\bar{n}$ $k \in R$ or $\bar{l} \times \bar{n} = \theta$, then the line is perpendicular to the plane.

(c) In all other cases, the angle θ between the line and the plane is given by



$$\theta = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \cos^{-1}\frac{|\bar{l}\cdot\bar{n}|}{|\bar{l}||\bar{n}|}$$

$$\therefore \quad \theta = \sin^{-1}\frac{|\bar{l}\cdot\bar{n}|}{|\bar{l}||\bar{n}|}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

(12) Common Section of Planes

Let $(\bar{r}-\bar{a})\cdot\bar{n}_1 = 0$ and $(\bar{r}-\bar{a})\cdot\bar{n}_2 = 0$ be two non-parallel planes having a common point $\bar{a}$. Then $(\bar{r}-\bar{a}) \perp \bar{n}_1$ and $(\bar{r}-\bar{a}) \perp \bar{n}_2$. So $\bar{r}-\bar{a}$ is parallel to $\bar{n}_1 \times \bar{n}_2 = \bar{n}$. Hence, $\bar{r}-\bar{a} = k\bar{n}$, $k \in R$.

$$\therefore \quad \bar{r} = \bar{a} + k\bar{n}, \quad k \in R$$

is the equation of the line passing through the common point $A(\bar{a})$ of the two planes in the direction of their common section. Common points of the two planes are on this line. Also every point on this line does satisfy the equations of both the planes.

Hence, $\bar{r} = \bar{a} + k\bar{n}$, $k \in R$ is the equation of common section of the two planes $(\bar{r}-\bar{a})\cdot\bar{n}_1 = 0$ and $(\bar{r}-\bar{a})\cdot\bar{n}_2 = 0$ where $\bar{n} = \bar{n}_1 \times \bar{n}_2$.