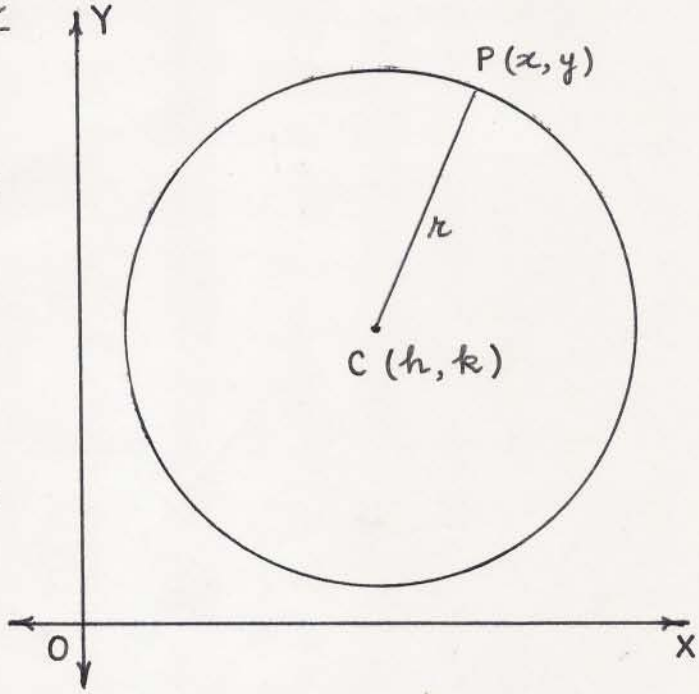
4. CIRCLE

4. CIRCLE

(1) Cartesian and Standard Equations of a Circle



The set of all points in the plane, which are at a constant distance from fixed point, is a circle. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

Let $C(h, k)$ be the centre of the circle and r its radius. Let $P(x, y)$ be any point on the circle. Then

$$CP = r \Leftrightarrow CP^2 = r^2$$
$$\Leftrightarrow (x-h)^2 + (y-k)^2 = r^2.$$

This is the cartesian equation of the circle with centre at $C(h, k)$ and radius, r.

If the centre is at origin and the radius is r, then the equation of the circle is

$$(x-0)^2 + (y-0)^2 = r^2, \quad \text{i.e.,} \quad x^2 + y^2 = r^2.$$

This is called the standard equation of the circle.

Further, if $r = 1$, then the circle is called the unit circle and its equation is $x^2 + y^2 = 1$.

(2) Quadratic and General Equations of a circle

The equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$

i.e., $x^2+y^2-2hx-2ky+h^2+k^2-r^2=0$

The general quadratic equation in R^2 is $Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$, where $A^2+H^2+B^2 \neq 0$. Comparing this with the above equation, it represents a circle if $H=0$ ... (i) and $A=B\neq 0$ ... (ii). Then, it takes the form

$$Ax^2+Ay^2+2Gx+2Fy+C=0$$

i.e., $$x^2+y^2+2\frac{G}{A}x+2\frac{F}{A}y+\frac{C}{A}=0$$

or, $x^2+y^2+2gx+2fy+c=0$, where $g=\frac{G}{A}$, $f=\frac{F}{A}$, $c=\frac{C}{A}$.

$\therefore\ x^2+2gx+g^2+y^2+2fy+f^2=g^2+f^2-c$

$\therefore\ (x+g)^2+(y+f)^2=g^2+f^2-c$

where $(-g,-f)$ is the centre and $\sqrt{g^2+f^2-c}$ is the radius of the circle.

For radius to be real $g^2+f^2-c>0$, i.e.,

$\left(\frac{G}{A}\right)^2+\left(\frac{F}{A}\right)^2-\left(\frac{C}{A}\right)>0$, or $G^2+F^2-AC>0$ ...(iii)

Thus (i), (ii) and (iii) are necessary and sufficient conditions that the general quadratic equation represents a circle. $x^2+y^2+2gx+2fy+c=0$ is called the general equation of a circle.

(3) Parametric Equations of a Circle

$$(x-h)^2+(y-k)^2=r^2 \Leftrightarrow \left(\frac{x-h}{r}\right)^2+\left(\frac{y-k}{r}\right)^2=1.$$

So, we can find $\theta \in (-\pi, \pi]$ for which

$$\frac{x-h}{r}=\cos\theta \quad \text{and} \quad \frac{y-k}{r}=\sin\theta$$

$\therefore\ x=h+r\cos\theta$ and $y=k+r\sin\theta$ represent parametric equations of a circle with centre (h, k) and radius, r where $\theta \in (-\pi, \pi]$ is the parameter.

If the centre is at origin, then the equations are $x = r\cos\theta$ and $y = r\sin\theta$. If $r = 1$, then we get $x = \cos\theta$ and $y = \sin\theta$ as the equations of the unit circle.

(4) Equation of a Circle given Diametrically Opposite Points

Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are diametrically opposite points of a circle. So the centre of the circle is $C\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

$\therefore$ the equation of the circle is

$$\left(x-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2=\left(x_1-\frac{x_1+x_2}{2}\right)^2+\left(y_1-\frac{y_1+y_2}{2}\right)^2$$

$$\Leftrightarrow \left(x-\frac{x_1+x_2}{2}\right)^2-\left(x_1-\frac{x_1+x_2}{2}\right)^2+\left(y-\frac{y_1+y_2}{2}\right)^2-\left(y_1-\frac{y_1+y_2}{2}\right)^2=0$$

$$\therefore\ (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

which is the required equation of the circle.

(5) Intersection of a Circle and a line

The line $ax+by+c=0$ intersects the circle $x^2+y^2=r^2$ in two, one or no point according as the perpendicular from the centre $(0,0)$ of the circle to the line given by $p=\frac{|c|}{\sqrt{a^2+b^2}}$ is smaller than, equal to or greater than radius, r of the circle, i.e., according as $\frac{c^2}{a^2+b^2}<r^2$, $\frac{c^2}{a^2+b^2}=r^2$ or, $\frac{c^2}{a^2+b^2}>r^2$.

Similar analysis can be done even for the general equation of a circle.

(6) Tangent and Normal to a Circle

Tangent : Suppose (x_1, y_1) is a point on the circle $x^2+y^2=r^2$. Differentiating the equation, we get, $2x + 2y\frac{dy}{dx} = 0$. Then $\frac{dy}{dx} = -\frac{x}{y}$.

$\therefore$ the slope of the tangent at the point (x_1, y_1) is

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{x_1}{y_1}, \quad y_1 \neq 0$$

$\therefore$ its equation is $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$.

$\therefore\ y_1 y - y_1^2 = -x_1 x + x_1^2$

$\therefore\ x_1 x + y_1 y = x_1^2 + y_1^2 = r^2$ [$\because (x_1, y_1)$ is on the circle.]

Thus, the equation of the tangent to the circle at the point (x_1, y_1) is $x_1 x + y_1 y = r^2$.

If $y_1 = 0$, then from $x_1^2 + y_1^2 = r^2$, we get $x_1 = \pm r$. The equations of the tangents at the points $(r, 0)$ and $(-r, 0)$ are $x = r$ and $x = -r$ respectively. The same equations are obtained if we put $(x_1, y_1) = (\pm r, 0)$ in the equation $x_1 x + y_1 y = r^2$.

Normal : The slope of the tangent at (x_1, y_1) is $-\frac{x_1}{y_1}$. Hence, if $x_1 \neq 0$, the slope of the normal at (x_1, y_1) is $\frac{y_1}{x_1}$. So, its equation is

$y - y_1 = (y_1/x_1)(x - x_1)$ or, $x_1 y - x_1 y_1 = y_1 x - x_1 y_1$

$\therefore\ y_1 x - x_1 y = 0$. For $x_1 = 0$, the equation of normal is $x = 0$ can also be obtained from this equation.

4. CIRCLE

(7) The Condition for the Line $y = mx + c$, to be Tangent to the Circle $x^2 + y^2 = r^2$ and the Point of Contact

Let $P(x_1, y_1)$ be the point of contact. The equation of tangent at P to the circle $x^2+y^2=r^2$ is

$$x_1 x + y_1 y - r^2 = 0 \quad \ldots \quad \ldots (1)$$

If this equation represents the same line as

$$mx - y + c = 0 \quad \ldots \quad \ldots \quad \ldots (2)$$

then $\dfrac{x_1}{m} = \dfrac{y_1}{-1} = \dfrac{-r^2}{c} \qquad \therefore \; x_1 = -\dfrac{r^2 m}{c}, \; y_1 = \dfrac{r^2}{c}$

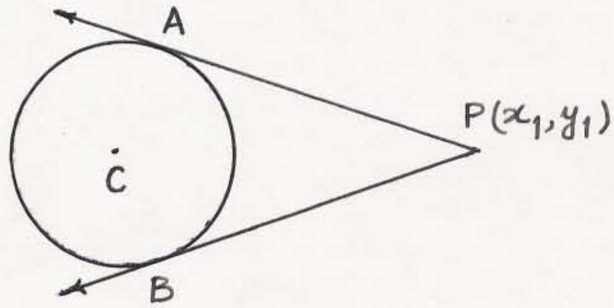
As $P(x_1, y_1)$ is on the circle, $x_1^2 + y_1^2 = r^2$.

$$\therefore \; \frac{r^4 m^2}{c^2} + \frac{r^4}{c^2} = r^2 \qquad \therefore \; r^2(1+m^2) = c^2$$

Hence, condition that the line $y = mx+c$ may be tangent to the circle $x^2+y^2=r^2$ is $r^2(1+m^2) = c^2$. The two tangents having slope m are $y = mx \pm r\sqrt{1+m^2}$. The coordinates of the points of contact are

$$\left(-\frac{r^2 m}{c}, \; \frac{r^2}{c}\right) \quad \text{i.e.,} \quad \left(\mp \frac{rm}{\sqrt{1+m^2}}, \; \pm \frac{r}{\sqrt{1+m^2}}\right)$$

(8) Tangents to the Circle from an Outside Point



Let $P(x_1, y_1)$ be a point outside the circle $x^2 + y^2 = r^2$.

$$\therefore \; x_1^2 + y_1^2 - r^2 > 0 \quad \ldots (1)$$

The equations of tangents to the circle having slope m are $y = mx \pm r\sqrt{1+m^2}$. If $P(x_1, y_1)$ is on it, then

$$y_1 = mx_1 \pm r\sqrt{1+m^2}$$

$$\therefore\ (y_1 - mx_1)^2 = r^2(1+m^2)$$

$$\therefore\ (r^2 - x_1^2)m^2 + 2x_1y_1m + r^2 - y_1^2 = 0 \quad \text{... (2)}$$

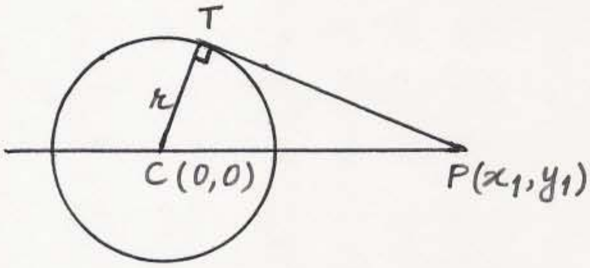
The discriminant of this quadratic equation in m is

$$\Delta = 4x_1^2y_1^2 - 4(r^2-x_1^2)(r^2-y_1^2)$$

$$= 4r^2(x_1^2+y_1^2-r^2) > 0 \text{ [by (1)]}$$

So equation (2) has two distinct real roots. Hence, two tangents can be drawn to a circle from a point outside the circle.

(9) Length of a Tangent



Let $\overline{PT}$ be the tangent from an external point $P(x_1, y_1)$ to the circle $x^2+y^2=r^2$.

The radius of the circle $CT = r$ and the centre is $C(0,0)$. $\therefore\ CP^2 = x_1^2 + y_1^2$

Now, in ΔPTC, $\overline{CT} \perp \overline{PT}$.

$$\therefore\ PT^2 + CT^2 = CP^2$$

$$\therefore\ PT^2 = CP^2 - CT^2 = x_1^2 + y_1^2 - r^2.$$

$\therefore$ the length of the tangent $PT = \sqrt{x_1^2+y_1^2-r^2}$.

It can similarly be proved that the length of the tangent from an external point $P(x_1, y_1)$ to the general circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$.

4. CIRCLE

(10) THEOREM : If two circles, $S_1 : x^2+y^2+2g_1x+2f_1y+c_1=0$ and $S_2 : x^2+y^2+2g_2x+2f_2y+c_2=0$ intersect in two distinct points, then the equation of the line containing their common chord is $S_1-S_2=0$.

Let $P(x_1, y_1)$ be one of the points of intersection.

Then, $x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1=0$ ... (1) and

$x_1^2+y_1^2+2g_2x_1+2f_2y_1+c_2=0$ ....(2)

From (1) and (2), $2(g_1-g_2)x_1+2(f_1-f_2)y_1+c_1-c_2=0$

Similarly, for other point of intersection, $Q(x_2, y_2)$,

$2(g_1-g_2)x_2+2(f_1-f_2)y_2+c_1-c_2=0$

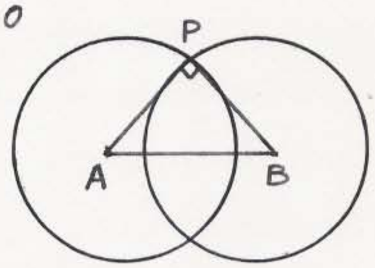
$\therefore$ $P(x_1, y_1)$ and $Q(x_2, y_2)$ satisfy the equation

$$2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0 \quad \text{... (3)}$$

As the circles are not concentric, $(g_1-g_2)^2+(f_1-f_2)^2 \neq 0$.

So, (3) represents a line in R^2 containing the common chord.

(11) Orthogonal Circles



If the circles $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$ are orthogonal and one of the points of intersection is P and if their centres are A and B respectively, then

$PA=\sqrt{g_1^2+f_1^2-c_1}$, $PB=\sqrt{g_2^2+f_2^2-c_2}$ & $AB=\sqrt{(g_1-g_2)^2+(f_1-f_2)^2}$

$\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ are tangents to these circles at P.

Now, in the right-angled ΔAPB, $\overline{PA} \perp \overline{PB}$.

$\therefore$ $PA^2+PB^2=AB^2$

$\therefore$ $g_1^2+f_1^2-c_1+g_2^2+f_2^2-c_2=(g_1-g_2)^2+(f_1-f_2)^2$

$\therefore$ $2g_1g_2+2f_1f_2=c_1+c_2$

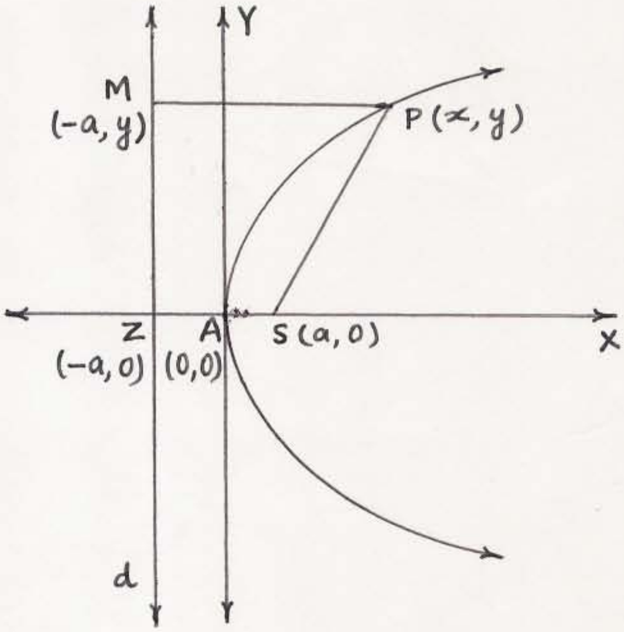
This is the necessary and sufficient condition for the given circles to be orthogonal.

5. PARABOLA

(1) Standard Equation of a Parabola

"The set of all points in the plane which are equidistant from a fixed line and a fixed point not on the line is called a parabola."

The fixed point is called the focus and the fixed line the directrix of the parabola.



Let d be a fixed line in a plane and S be a fixed point not on it. Let $\overline{SZ}$ be perpendicular to d meeting d in Z. The mid-point A of $\overline{SZ}$ is equidistant from S and d. So it lies on the parabola. Let A be the origin, $\overleftrightarrow{AS}$ the X-axis and $\overrightarrow{AS}$ the positive direction of X-axis. If S is the point $(a, 0)$, then Z will be $(-a, 0)$. Hence, the equation of the directrix is $x = -a$.

Let $P(x, y)$ be any point on the parabola. Then it must be equidistant from S and d. $\overline{PM}$ is drawn perpendicular to d. Then the coordinates of M will be $(-a, y)$.

Now, $SP = PM \quad \therefore \ SP^2 = PM^2$

$\therefore \quad (x-a)^2 + y^2 = (x+a)^2$

$\therefore \quad y^2 = (x+a)^2 - (x-a)^2$

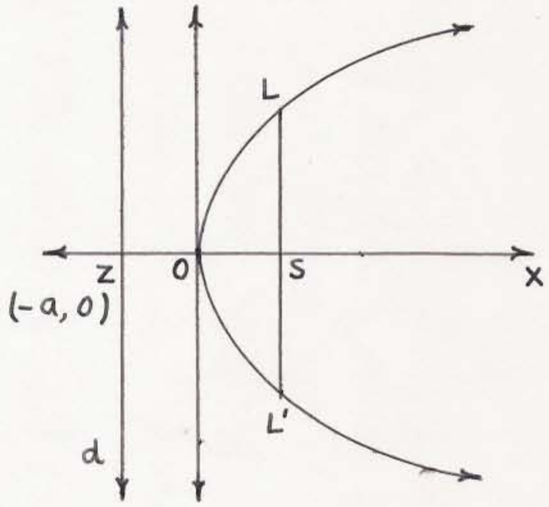
$\therefore \quad y^2 = 4ax.$

This is the standard equation of a parabola. The line through the focus perpendicular to directrix is the axis of the parabola. The point of intersection of the parabola with its axis is called the vertex of the parabola.

(2) Parametric Equations of a Parabola

$x = at^2$, $y = 2at$, $t \in R$ are the parametric equations of a parabola and the point $(at^2, 2at)$ is called the t-point of the parabola. Eliminating t from these equations, we get $y^2 = 4ax$.

(3) Latus - Rectum of a Parabola



The chord of a parabola passing through its focus is called a focal-chord of a parabola and if it is perpendicular to its axis, it is called the latus-rectum of the parabola.

If L and L' are end-points of the latus-rectum, then $\overline{LL'}$ is perpendicular to the X-axis. Hence, the perpendicular distance of L from the directrix = SZ. But L is on parabola. Hence, its distance from the focus, LS = its distance from the directrix, SZ.

Now, LL' = 2LS [$\because$ parabola is symmetric with respect to its axis]

$= 2SZ$

$= 2\,|2a|$ [for the parabola $y^2 = 4ax$]

$= 4|a|$

The coordinates of L and L' are $(a, 2a)$ and $(a, -2a)$.

(4) The Equation of the Tangent at the point (x_1, y_1) of the Parabola $y^2 = 4ax$

Differentiating the equation of the parabola $y^2 = 4ax$ with respect to x, we get

$$2y\frac{dy}{dx} = 4a \qquad \text{or,} \quad \frac{dy}{dx} = \frac{2a}{y}$$

So, except at $(0,0)$, the slope of the tangent at (x_1, y_1)

$$\text{is } \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2a}{y_1}$$

Therefore the equation of the tangent is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

$$\text{i.e.,} \quad yy_1 - y_1^2 = 2ax - 2ax_1$$

But the point (x_1, y_1) is on the parabola. $\therefore\ y_1^2 = 4ax_1$.
Hence, the equation of the tangent at (x_1, y_1) is

$$yy_1 = 2a(x + x_1).$$

At the origin $(0,0)$, the tangent to the parabola is the Y-axis which is obtained by putting $(x,y) = (0,0)$ in the above equation.

(5) The Equation of the tangent at the point t to the Parabola $y^2 = 4ax$

The parametric equations of the parabola are

$$x = at^2, \quad y = 2at, \quad t \in R.$$

Differentiating with respect to t, $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = 2a\cdot\frac{1}{2at} = \frac{1}{t} \quad (\text{for } t \neq 0).$$

Thus, the slope of the tangent at point t ($\neq 0$) is $\frac{1}{t}$ and the coordinates of the point of contact are $(at^2, 2at)$.

$\therefore$ Equation of the tangent is $y - 2at = \frac{1}{t}(x - at^2)$

i.e., $ty = x + at^2$.

At the point corresponding to $t=0$, i.e., at $(0,0)$, the tangent to the parabola is Y-axis which is obtained by putting $t=0$ in the above equation.

(6) The Condition for the Line $y = mx + c$ to be a Tangent to the Parabola $y^2 = 4ax$ and the Coordinates of the point of contact

The equation of the tangent to the parabola $y^2 = 4ax$ at (x_1, y_1) is $y_1 y = 2a(x + x_1)$ ... (1).

If the line $y = mx + c$ ... (2) is the tangent to the parabola at (x_1, y_1), then (1) and (2) represent the same line.

$$\therefore \frac{y_1}{1} = \frac{2a}{m} = \frac{2ax_1}{c}$$

$$\therefore (x_1, y_1) = \left(\frac{c}{m}, \frac{2a}{m}\right)$$

But (x_1, y_1) is a point on the parabola.

$$\therefore y_1^2 = 4ax_1 \quad \text{or,} \quad \frac{4a^2}{m^2} = \frac{4ac}{m} \quad \text{or,} \quad c = \frac{a}{m}$$

which is the necessary and sufficient condition for $y = mx + c$ to be a tangent of the parabola $y^2 = 4ax$ and the coordinates of the point of contact are

$$(x_1, y_1) = \left(\frac{c}{m}, \frac{2a}{m}\right) = \left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

Also, the equation of the tangent with slope m to the parabola $y^2 = 4ax$ is $y = mx + a/m$.

(7) Tangents to the Parabola from a Given Point

Let m = slope of the tangent to the parabola from a given point (x_1, y_1).

Then, $y = mx + a/m$ is the equation of this tangent.

Now, (x_1, y_1) lies on this tangent.

$$\therefore \; y_1 = mx_1 + \frac{a}{m}$$

$$\text{or, } m^2 x_1 - my_1 + a = 0.$$

This is a quadratic equation in m if $x_1 \neq 0$. If the discriminant of this quadratic equation in m,

$\Delta = y_1^2 - 4ax_1 > 0$, then there are two values of m and two tangents to the parabola from (x_1, y_1).

If $y_1^2 - 4ax_1 = 0$, then only one tangent can be drawn to the parabola corresponding to one value of m and (x_1, y_1) is a point on the parabola.

If $y_1^2 - 4ax_1 < 0$, then no tangent can be drawn to the parabola from (x_1, y_1).

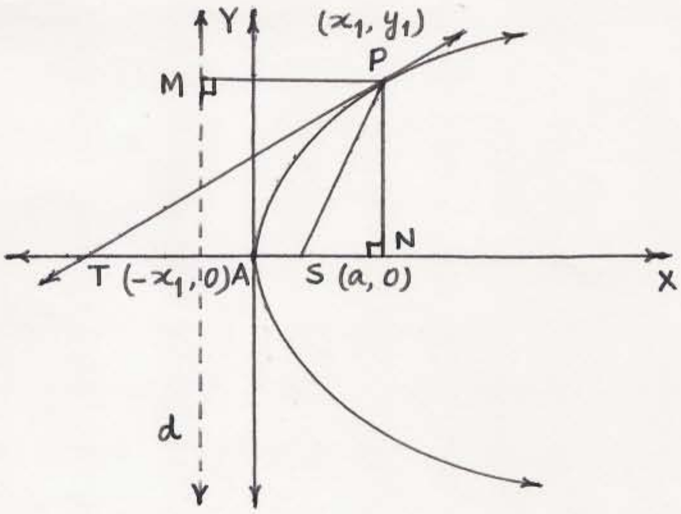
(8) Properties of Parabola

(i) If the tangent at a point P on the parabola meets the axis of the parabola in T and if S is the focus, then ST = SP

Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 4ax$ whose axis is the X-axis and focus is $S(a, 0)$.

The equation of the tangent at P is $y_1 y = 2a(x + x_1)$ ... (1)

The point of intersection of line (1) with the X-axis



is $T(-x_1, 0)$.

$\therefore \quad ST = |a + x_1|$

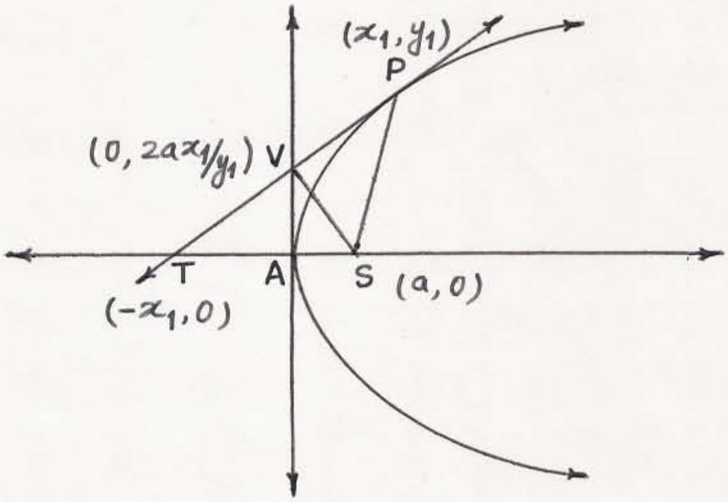
Also, $SP = PM = NZ$ where M and N are feet of perpendiculars from P on the directrix and the X-axis respectively.

$\therefore \quad SP = NZ = NA + AZ$

$\qquad = |a + x_1|$

$\therefore \quad ST = SP.$

(ii) The foot of the perpendicular from the focus to any tangent of the parabola lies on the tangent at the vertex



Let $P(x_1, y_1)$ be a point on the parabola, $y^2 = 4ax$ whose axis is the X-axis, focus is $S(a, 0)$ and the vertex is $A(0, 0)$. The tangent at the vertex A is the Y-axis, i.e., $x = 0$ ...(1). The equation of the tangent at P is $y_1 y = 2a(x + x_1)$ ... (2). The points of intersection of the tangents (1) and (2) is $V(0, 2ax_1/y_1)$. The point of intersection of the tangent at P with X-axis is $T(-x_1, 0)$.

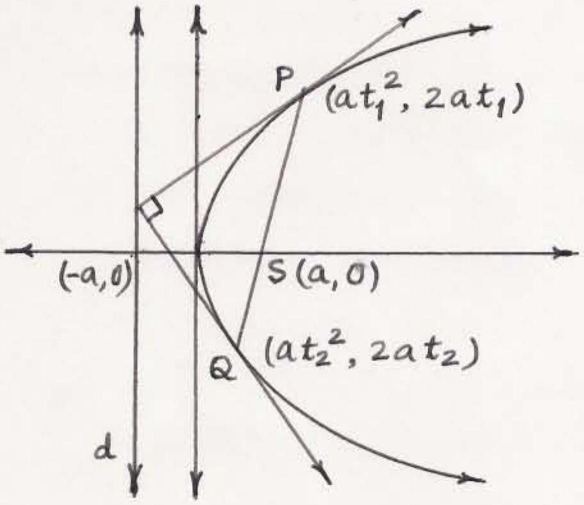
$$\therefore \quad TV = \sqrt{x_1^2 + 4a^2x_1^2/y_1^2} = \sqrt{x_1^2 + ax_1} \quad (\because y_1^2 = 4ax_1)$$

$$\text{and } TP = \sqrt{4x_1^2 + y_1^2} = \sqrt{4x_1^2 + 4ax_1} = 2\,TV$$

Thus, the correspondence PSV $\leftrightarrow$ TSV is a congruence. (SSS – theorem). $\therefore$ $m\angle SVP = m\angle SVT = 90°$

($\because m\angle SVP + m\angle SVT = 180°$).

(iii) The tangents at the end points of a focal-chord of parabola intersect orthogonally at the directrix.



Let $P(t_1)$ and $Q(t_2)$ be the end-points of the focal chord of the parabola, $y^2 = 4ax$ whose focus is $S(a,0)$. The points P, Q and S are collinear.

$$\therefore \begin{vmatrix} a & 0 & 1 \\ at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \end{vmatrix} = 0$$

$\therefore$ $2a^2(t_1 - t_2)(1 + t_1t_2) = 0$.

$\therefore$ $1 + t_1t_2 = 0$ or, $t_1t_2 = -1$ ($\because a \neq 0,\ t_1 \neq t_2$).

Now, the equations of the tangents at P and Q are

$t_1 y = x + at_1^2$ and

$t_2 y = x + at_2^2$, respectively.

The coordinates of their point of intersection are

$[at_1t_2,\ a(t_1 + t_2)]$.

But, $t_1t_2 = -1$. $\therefore$ the x-coordinate is $-a$.

$\therefore$ the point of intersection is on the directrix.

Also, the slopes of the tangents are $\frac{1}{t_1}$ and $\frac{1}{t_2}$. Since $t_1t_2 = -1$, they are orthogonal to each other.

(iv) For the parabola $y^2 = 4ax$ if S is the focus and $\overline{PQ}$ is the focal chord, then $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$

If the end-points of the focal chord are $P(t_1)$ and $Q(t_2)$, then $t_1 t_2 = -1$.

Now, $SP^2 = (at_1^2 - a)^2 + (2at_1 - 0)^2$

$= a^2 (t_1^2 + 1)^2.$

and $SQ^2 = a^2 (t_2^2 + 1)^2$

$$= a^2 \left(\frac{1}{t_1^2} + 1\right)^2$$

$$= \frac{a^2 (t_1^2 + 1)^2}{t_1^4}.$$

$$\therefore \frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a(t_1^2+1)} + \frac{t_1^2}{a(t_1^2+1)} = \frac{1}{a}.$$

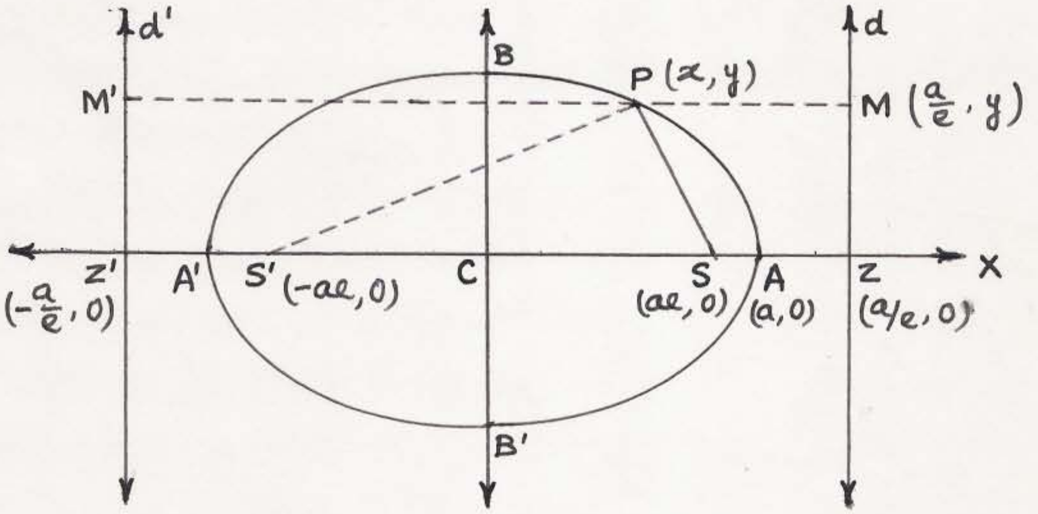
6. ELLIPSE

6. ELLIPSE

(1) Standard Equation of an ellipse

"The set of all points in a plane, whose distance from a fixed point is in a constant ratio e to its distance from a fixed line not containing the fixed point, is called an ellipse if e is less than 1."

The fixed point is called the focus and the fixed line is called the directrix of the ellipse.



Let d be a fixed line in a plane and S a fixed point not on it. Let $\overline{SZ}$ be perpendicular to d meeting d in Z. Let A be a point on $\overleftrightarrow{SZ}$, such that S–A–Z and $\frac{SA}{AZ} = e$, and also A' be a point such that A'–S–Z and $\frac{SA'}{A'Z} = e$. Obviously, A and A' are points on the ellipse, and they divide $\overline{SZ}$ in the ratio e and –e respectively.

Let C be the mid-point of $\overline{AA'}$ and let $AA' = 2a$. C is taken as the origin, $\overleftrightarrow{CZ}$ as the X-axis and $\overrightarrow{CZ}$ as the positive direction of X-axis. Then A is the point (a, 0) and $A'(-a, 0)$. Suppose S is $(p, 0)$ and $Z(q, 0)$. As A divides $\overline{SZ}$ in the ratio e, we have

$$a = \frac{eq + p}{e + 1}, \quad \text{i.e.,} \quad ae + a = eq + p \quad \text{... (i)}$$

Also, as A' divides $\overline{SZ}$ in the ratio $-e$, we have

$$-a = \frac{-eq + p}{-e + 1}, \quad \text{i.e.,} \quad ae - a = -eq + p \quad \text{... (ii)}$$

From (i) and (ii), we get $p = ae$ and $q = \frac{a}{e}$.
Thus, the coordinates of focus S are $(ae, 0)$ and the equation of the directrix is $x = a/e$.
Let $P(x, y)$ be any point on the ellipse. $\overline{PM}$ is drawn perpendicular to d, so that M is the point $(\frac{a}{e}, y)$.
Now, $SP = e.PM \quad \therefore SP^2 = e^2.PM^2$

$$\therefore (x - ae)^2 + y^2 = e^2(a/e - x)^2$$

$$\therefore x^2 - 2aex + a^2e^2 + y^2 = x^2e^2 - 2aex + a^2 \quad \text{... (iii)}$$

$$\therefore x^2(1 - e^2) + y^2 = a^2(1 - e^2)$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

Here, $e < 1$ and so $e^2 < 1$. Hence $a^2(1 - e^2) > 0$.
Putting $a^2(1 - e^2) = b^2$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$ which is the standard equation of the ellipse.

Adding $4aex$ to both sides of equation (iii),

$$(x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2.$$

i.e., if M' is the foot of perpendicular from P to $d' : x = -\frac{a}{e}$ and S' is the point $(-ae, 0)$, then $S'P^2 = e^2 PM'^2$. Thus, there are two focus-directrix pairs for the ellipse and its equation can be derived with respect to any one pair.

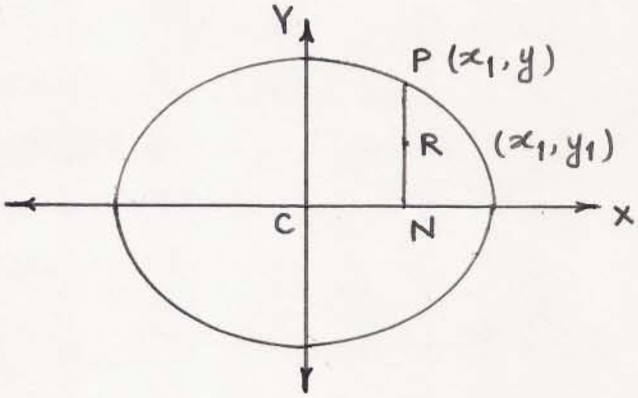
6. ELLIPSE

(2) Parametric Equations of an Ellipse

$x = a\cos\theta$ and $y = b\sin\theta$, $\theta \in (-\pi, \pi]$ are the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Eliminating θ gives the equation of the ellipse. The point $(a\cos\theta, b\sin\theta)$ is called the θ-point of the ellipse.

(3) The position of the Point $R(x_1, y_1)$ relative to an Ellipse

Let $R(x_1, y_1)$ be inside the ellipse. Let N be the foot of perpendicular from R to the major axis. Suppose $\overrightarrow{NR}$ intersects the ellipse in the point $P(x_1, y)$.



$\therefore\ y_1^2 < y^2 \qquad \therefore\ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < \frac{x_1^2}{a^2} + \frac{y^2}{b^2} = 1$

($\because$ $P(x_1, y)$ is on the ellipse)

$\therefore\ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0$. The converse is also true.

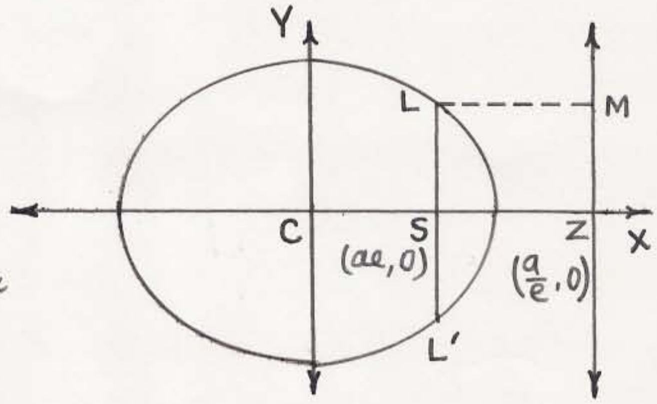
Thus, if $S_1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$, then

$S_1 < 0 \iff R(x_1, y_1)$ is inside the ellipse.
$S_1 = 0 \iff R(x_1, y_1)$ is on the ellipse.
$S_1 > 0 \iff R(x_1, y_1)$ is outside the ellipse.

(4) Latus-Rectum of an Ellipse

The chord of an ellipse passing through its focus is called the focal chord of the ellipse and if it is perpendicular to the major axis, it is called the latus-rectum.

$\overline{LL'}$ is the latus-rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ passing through the focus $S(ae, 0)$. M is the foot of perpendicular from L to the directrix corresponding to S.



Now, $SL = e.LM = e.SZ$ ($\because$ □ LMZS is a rectangle)

$$= e\left(\frac{a}{e}-ae\right) = a(1-e^2) = \frac{b^2}{a}$$

$\therefore$ length of the latus-rectum, $LL' = \frac{2b^2}{a}$ (if $a>b$)

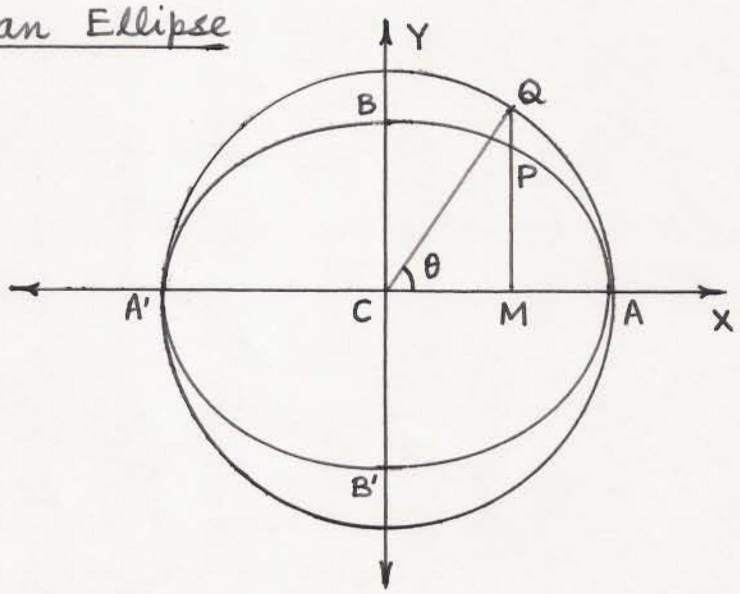
and $LL' = \frac{2a^2}{b}$ (if $b>a$).

(5) Auxiliary Circle of an Ellipse

"The circle whose diameter is the major axis of the ellipse is called the auxiliary circle of the ellipse."

Its equation is

$$x^2+y^2=a^2.$$



Suppose the parametric coordinates of a point P on the ellipse are $(a\cos\theta, b\sin\theta)$. M is the foot of perpendicular from P to the X-axis and $\overrightarrow{MP}$ intersects the auxiliary circle in Q.

$\therefore$ the x-coordinate of Q is $a\cos\theta$.

But the equation of the auxiliary circle is $x^2+y^2=a^2$.
$\therefore$ the y-coordinate of Q is $a\sin\theta$.
$\therefore$ Q is $(a\cos\theta,\ a\sin\theta)$
θ is called the eccentric angle of the point P. P and Q are called coherent points of the ellipse and the circle respectively.

(6) Equation of the Tangent at the Point $P(x_1, y_1)$ of the Ellipse $x^2/a^2 + y^2/b^2 = 1$

Differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with respect to x, $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$.

The slope of the tangent at the point $P(x_1, y_1)$ is

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{b^2x_1}{a^2y_1} \qquad [\text{for } y_1 \neq 0]$$

$\therefore$ the equation of the tangent at P is

$$y-y_1 = -\frac{b^2x_1}{a^2y_1}(x-x_1),$$

i.e., $$\frac{y_1y}{b^2} - \frac{y_1^2}{b^2} = -\frac{x_1x}{a^2} + \frac{x_1^2}{a^2},$$

or $$\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \qquad [\because P(x_1,y_1) \text{ is on the ellipse}]$$

This equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is also valid for $y_1=0$. The points on the ellipse corresponding to $y_1=0$ are $(\pm a, 0)$. The equations of tangents to the ellipse, obtained by putting $x_1=\pm a$ and $y_1=0$ in the above equation, are $x=\pm a$.

6. ELLIPSE

(7) The Equation of Tangent at the point $P(\theta)$ of the Ellipse $x^2/a^2 + y^2/b^2 = 1$

Substituting $(x_1, y_1) = (a\cos\theta, b\sin\theta)$ in the equation of tangent $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ which is the required equation.

(8) Condition for the Line $y = mx + c$, $c \neq 0$ to be a Tangent of the Ellipse and the Coordinates of the Point of Contact

Let $y = mx + c$ be tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point $P(x_1, y_1)$ for $m \neq 0$.
Now, equation of tangent to the ellipse at $P(x_1, y_1)$ is $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} - 1 = 0$ ... (1) which must be same as $mx - y + c = 0$ ...(2).

Comparing equations (1) and (2), $\frac{x_1}{a^2 m} = \frac{y_1}{-b^2} = \frac{-1}{c}$

$\therefore\ x_1 = -\frac{a^2 m}{c}$ and $y_1 = \frac{b^2}{c}$. But $P(x_1, y_1)$ is on the ellipse. $\therefore\ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \Rightarrow \frac{a^4 m^2}{c^2 a^2} + \frac{b^4}{c^2 b^2} = 1.$

$\therefore\ c^2 = a^2 m^2 + b^2$ is the necessary condition and the coordinates of the point of contact $= (-a^2 m/c,\ b^2/c)$.
[valid for $m = 0$ also].

(9) Tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from $P(x_1, y_1)$

The equations of the tangents of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with slope m are $y = mx \pm \sqrt{a^2 m^2 + b^2}$.

If this tangent passes through the point $P(x_1, y_1)$, then

$$y_1 = mx_1 \pm \sqrt{a^2m^2+b^2}.$$

$$\therefore \quad (y_1 - mx_1)^2 = a^2m^2 + b^2$$

$$\therefore \quad (a^2 - x_1^2)\,m^2 + 2x_1y_1m + b^2 - y_1^2 = 0$$

which is a quadratic equation in m. So, at the most two tangents can be drawn to the ellipse from a point P corresponding to two distinct real roots of m.

The discriminant of this equation

$$= 4x_1^2y_1^2 - 4(a^2-x_1^2)(b^2-y_1^2)$$

$$= 4a^2b^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right)$$

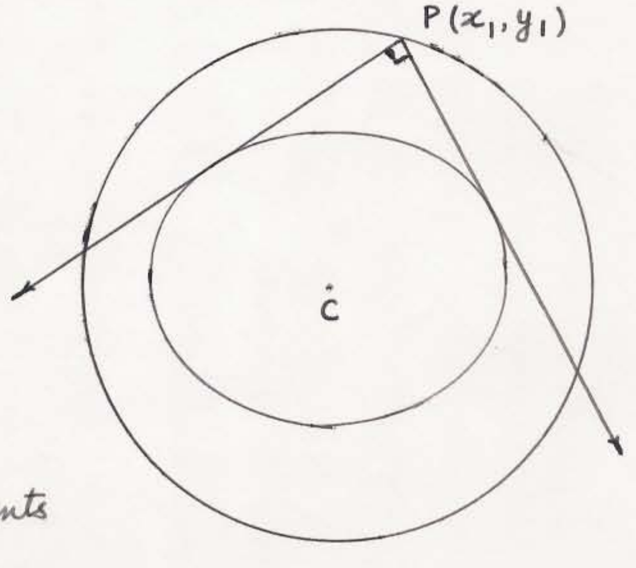
$$= 4a^2b^2S_1 \quad \text{where } S = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

(a) If $S_1 > 0$, two tangents can be drawn to the ellipse from (x_1, y_1). Here (x_1, y_1) is outside the ellipse.

(b) If $S_1 = 0$, then only one tangent can be drawn to the ellipse from (x_1, y_1). Here (x_1, y_1) lies on the ellipse.

(c) If $S_1 < 0$, then no tangent can be drawn to the ellipse from (x_1, y_1). Here (x_1, y_1) is inside the ellipse.

(10) Director Circle of Ellipse



Let us find the set of all points in the plane such that the tangents drawn from them to the ellipse are perpendicular to each other.

The equations of the tangents

of the ellipse $x^2/a^2 + y^2/b^2 = 1$ with slope m are

$$y = mx \pm \sqrt{a^2m^2+b^2}$$

If this tangent passes through the point $P(x_1, y_1)$, then

$$y_1 = mx_1 \pm \sqrt{a^2m^2+b^2}.$$

$$\therefore \; (y_1 - mx_1)^2 = a^2m^2 + b^2$$

$$\therefore \; (a^2 - x_1^2)m^2 + 2x_1y_1m + b^2 - y_1^2 = 0$$

The roots m_1, m_2 of this equation are the slopes of the tangents drawn to the ellipse from $P(x_1, y_1)$.

If $m_1m_2 = -1$, then $\dfrac{b^2-y_1^2}{a^2-x_1^2} = -1$

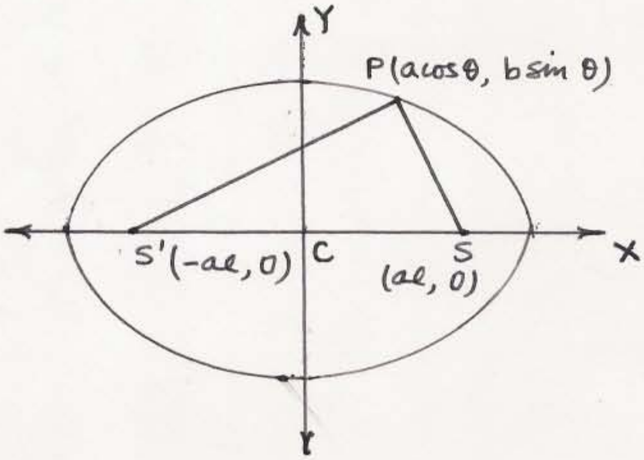
i.e., $x_1^2 + y_1^2 = a^2 + b^2$.

This shows that the point $P(x_1, y_1)$ lies on the circle $x^2 + y^2 = a^2 + b^2$, the tangents drawn from which to the ellipse $x^2/a^2 + y^2/b^2 = 1$ are perpendicular to each other. This circle is called the director circle of the ellipse.

(11) Properties of Ellipse

(i) If S and S′ are foci of an ellipse and P is any point on the ellipse, then $SP + S'P = 2a$ (a positive constant).

Let $P(a\cos\theta, b\sin\theta)$ be any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$. $S(ae, 0)$ and $S'(-ae, 0)$ are the foci.



6. ELLIPSE

$$\therefore \quad SP^2 = (ae - a\cos\theta)^2 + b^2\sin^2\theta$$
$$= a^2e^2 - 2a^2e\cos\theta + a^2\cos^2\theta + a^2(1-e^2)\sin^2\theta$$
$$= a^2e^2\cos^2\theta - 2a^2e\cos\theta + a^2$$
$$= a^2(1-e\cos\theta)^2$$

$$\therefore \quad SP = a(1-e\cos\theta) \quad (\because e<1, \cos\theta<1).$$

Similarly, it can be shown that $S'P = a(1+e\cos\theta)$.

$$\therefore \quad SP + S'P = 2a.$$

(ii) If S and S' are foci of an ellipse and B is its vertex on minor axis, then $SB = S'B = a$ (length of semi-major axis

$S(ae, 0)$ and $S'(-ae, 0)$ are the foci and $B(0, b)$ is a vertex on minor axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

$$\therefore \quad SB^2 = a^2e^2 + b^2 = a^2e^2 + a^2(1-e^2) = a^2$$

$$\therefore \quad SB = a \text{ (length of semi-major axis).}$$

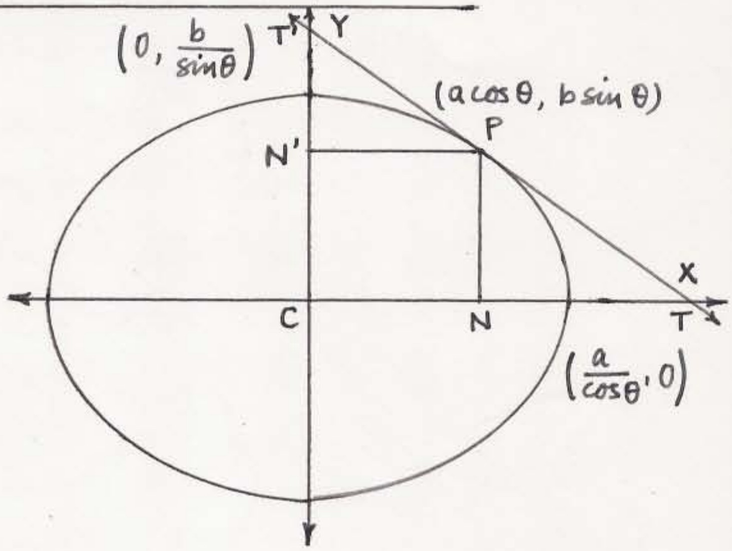
Similarly, $S'B = a$.

(iii) If S and S' are foci of an ellipse and A and A' are its vertices on the major axis, then $AS.A'S = b^2$, where b is the length of semi-minor axis.

$S(ae, 0)$ is a focus and $A(a, 0)$ and $A'(-a, 0)$ are the vertices on major axis of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

$$\therefore \quad AS.\,A'S = (a-ae)(a+ae)$$
$$= a^2(1-e^2) = b^2$$

(iv) C is the centre of the ellipse $x^2/a^2 + y^2/b^2 = 1$. The tangent at the point $P(a\cos\theta, b\sin\theta)$ intersect the major axis and minor axis in T and T'. N and N' are the feet of perpendiculars from P to the X- and Y-axis respectively. Then, (a) $CT.CN = a^2$, (b) $CT'.CN' = b^2$ and (c) $a^2/CT^2 + b^2/CT'^2 = 1$.



The equation of tangent at the point $P(a\cos\theta, b\sin\theta)$ of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Its points of intersection with the X- and Y-axis are $T(a/\cos\theta, 0)$ and $T'(0, b/\sin\theta)$ respectively.

The coordinates of the feet of perpendiculars from P on the X- and Y-axis are $N(a\cos\theta, 0)$ and $N'(0, b\sin\theta)$ respectively.

$\therefore$ (a) $CT.CN = \dfrac{a}{\cos\theta} \times a\cos\theta = a^2.$

(b) $CT'.CN' = \dfrac{b}{\sin\theta} \times b\sin\theta = b^2$

(c) $\dfrac{a^2}{CT^2} + \dfrac{b^2}{CT'^2} = \dfrac{a^2}{\frac{a^2}{\cos^2\theta}} + \dfrac{b^2}{\frac{b^2}{\sin^2\theta}}$

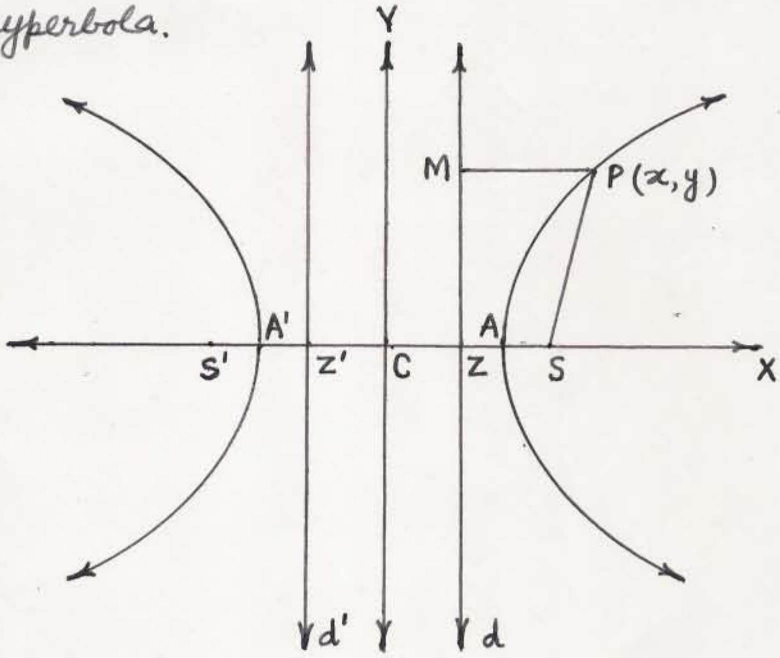
$= \cos^2\theta + \sin^2\theta$

$= 1.$

7. HYPERBOLA

(1) Standard Equation of a Hyperbola

" The set of all points in the plane, whose distance from a fixed point is in a constant ratio e to its distance from a fixed line not containing the fixed point, is called a hyperbola, if e is greater than 1."
The fixed point is called the focus, the fixed line is the directrix and the constant ratio is called the eccentricity of the hyperbola.



Let d be a fixed line in a plane and S a fixed point not on it. Let $\overline{SZ}$ be perpendicular to d meeting d in Z. Let A be a point on $\overleftrightarrow{SZ}$, such that S–A–Z and $\frac{SA}{AZ} = e$, and also A' be a point such that A'–Z–S and $\frac{SA'}{A'Z} = e$.
Obviously, A and A' are points on the hyperbola, and they divide $\overline{SZ}$ in the ratio e and $-e$ respectively.

Let C be the mid-point of $\overline{AA'}$ and let $AA' = 2a$. C is taken as the origin, $\overleftrightarrow{CZ}$ as the X-axis and $\overrightarrow{CZ}$ as the positive direction of X-axis. Then A is the point $(a, 0)$ and $A'(-a, 0)$.
Suppose S is $(p, 0)$ and Z is $(q, 0)$. As A divides $\overline{SZ}$

in the ratio e from S, we have

$$a = \frac{eq+p}{e+1}, \qquad \text{i.e.,} \quad ae + a = eq + p \quad \dots (i)$$

Also, as A' divides $\overline{SZ}$ in the ratio $-e$ from S, we have

$$-a = \frac{-eq+p}{-e+1}, \qquad \text{i.e.,} \quad ae - a = -eq + p \quad \dots (ii)$$

From (i) and (ii), we get $p = ae$ and $q = \frac{a}{e}$.
Thus, the coordinates of focus S are $(ae, 0)$ and the equation of the directrix is $x = a/e$.
Let $P(x, y)$ be any point on the hyperbola. $\overline{PM}$ is drawn perpendicular to d, so that M is the point $\left(\frac{a}{e}, y\right)$.
Now, $SP = e.PM \quad \therefore \quad SP^2 = e^2.PM^2$

$$\therefore \quad (x-ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\therefore \quad x^2 - 2aex + a^2e^2 + y^2 = e^2x^2 - 2aex + a^2 \quad \dots (iii)$$

$$\therefore \quad (e^2-1)x^2 - y^2 = a^2(e^2-1)$$

$$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

Here, $e > 1$ and so $e^2 > 1$. Hence $a^2(e^2-1) > 0$.
Putting $a^2(e^2-1) = b^2$, we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$ which is the standard equation of the hyperbola.

Adding $4aex$ to both sides of equation (iii),

$$(x+ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2,$$

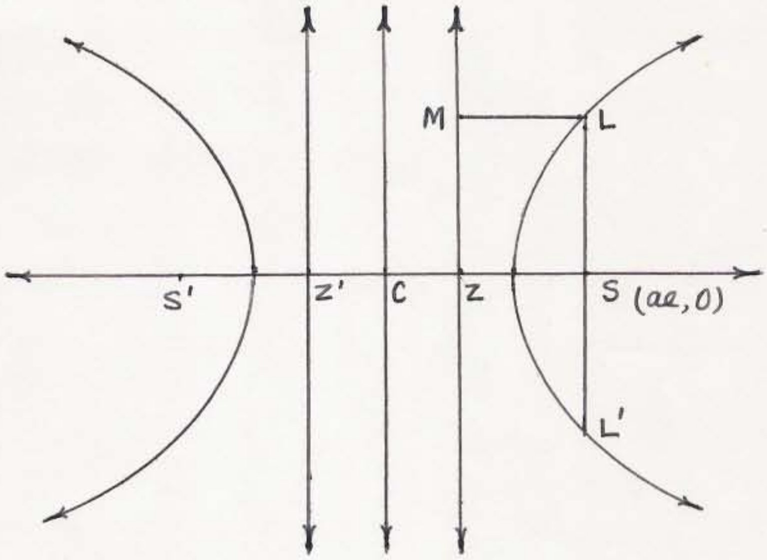
i.e., if M' is the foot of perpendicular from P to $d' : x = -\frac{a}{e}$ and S' is the point $(-ae, 0)$, then $S'P^2 = e^2 PM'^2$. Thus, there are two focus-directrix pairs for the hyperbola and its equation can be derived with respect to any one pair.

7. HYPERBOLA

(2) Parametric Equations of a Hyperbola

$x = a\sec\theta$ and $y = b\tan\theta$, $\theta \in (-\pi, \pi] - \{\pm\frac{\pi}{2}\}$ are the parametric equations of the hyperbola. Eliminating θ gives the equation of the hyperbola. The point $(a\sec\theta, b\tan\theta)$ is called the θ-point of the hyperbola.

(3) Latus-Rectum of a Hyperbola



The chord of the hyperbola passing through its focus is called the focal chord of the hyperbola and if it is perpendicular to the transverse axis, it is called the latus-rectum of the hyperbola.

$\overline{LL'}$ is a latus-rectum of the hyperbola.

Now, $SL = e.LM = e\,ZS$

$$= e\left(ae - \frac{a}{e}\right) = a(e^2-1) = \frac{b^2}{a}$$

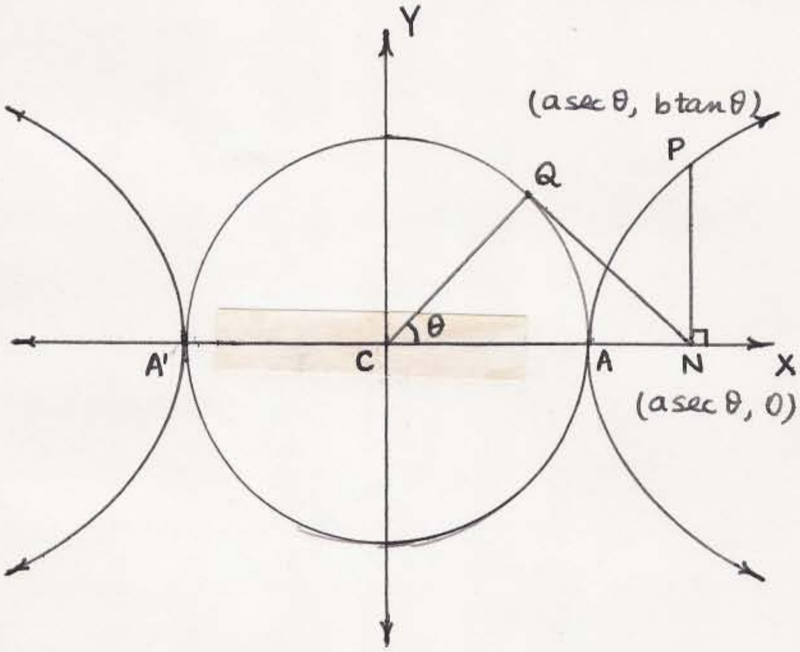
$$\therefore\ LL' = 2SL = \frac{2b^2}{a} \text{ for the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, $LL' = \frac{2a^2}{b}$.

(4) Auxiliary Circle of the Hyperbola

" The circle whose diameter is the transverse axis of the hyperbola is called the auxiliary circle of the hyperbola. For the hyperbola $x^2/a^2 - y^2/b^2 = 1$, its equation is $x^2 + y^2 = a^2$

Let $P(a\sec\theta, b\tan\theta)$ be any point on the hyperbola. Let N be the foot of perpendicular from P to the X-axis. Then, the coordinates of N are $(a\sec\theta, 0)$. The tangent from N to the auxiliary circle meets it at Q. Then, the coordinates of Q are $(a\cos\theta, a\sin\theta)$



$m\angle QCA = \theta$, $\theta \in (-\pi, \pi] - \{\pm \frac{\pi}{2}\}$, is called the eccentric angle of point P on the hyperbola.

(5) <u>Equation of the tangent at the Point $P(x_1, y_1)$ of the Hyperbola $x^2/a^2 - y^2/b^2 = 1$.</u>

Differentiating $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to x, $\frac{dy}{dx} = \frac{b^2x}{a^2y}$.

The slope of the tangent at the point $P(x_1, y_1)$ is

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{b^2x_1}{a^2y_1} \quad (\text{when } y_1 \neq 0)$$

$\therefore$ the equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

i.e., $$\frac{y_1y}{b^2} - \frac{y_1^2}{b^2} = \frac{x_1x}{a^2} - \frac{x_1^2}{a^2}$$

$\therefore \dfrac{x_1 x}{a^2} - \dfrac{y_1 y}{b^2} = \dfrac{x_1^2}{a^2} - \dfrac{y_1^2}{b^2} = 1$ [$\because P(x_1, y_1)$ is on the hyperbola]

This equation of tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is also valid for $y_1 = 0$. The points on the hyperbola corresponding to $y_1 = 0$ are $(\pm a, 0)$. The equations of tangents to the hyperbola, obtained by putting $x_1 = \pm a$ and $y_1 = 0$ in the above equation, are $x = \pm a$.

(6) The Equation of Tangent at the Point $P(\theta)$ of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Substituting $(x_1, y_1) = (a \sec\theta, b \tan\theta)$ in the equation of tangent $x_1 x/a^2 - y_1 y/b^2 = 1$ of the hyperbola $x^2/a^2 - y^2/b^2 = 1$, we get $x \sec\theta/a - y \tan\theta/b = 1$ which is the equation of the tangent at the point $P(\theta)$ of the hyperbola.

(7) Condition for the Line $y = mx + c$, $c \neq 0$ to be a Tangent of the Hyperbola and Coordinates of the Point of Contact

Let the line $y = mx + c$ be a tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point $P(\theta)$ for $m \neq 0$. Now, equation of tangent to the hyperbola at the point $P(\theta)$ is

$\dfrac{x \sec\theta}{a} - \dfrac{y \tan\theta}{b} - 1 = 0$ ... (i) which must be same as

$mx - y + c = 0$ ... (ii)

Comparing equations (i) and (ii), $\dfrac{\sec\theta}{am} = \dfrac{\tan\theta}{b} = -\dfrac{1}{c}$

$\therefore \sec\theta = -\dfrac{am}{c}$ and $\tan\theta = -\dfrac{b}{c}$.

Eliminating θ from these equations, $\dfrac{a^2 m^2}{c^2} - \dfrac{b^2}{c^2} = 1$.

$\therefore c^2 = a^2 m^2 - b^2$ is the necessary condition and point of contact is $(-a^2 m/c, -b^2/c)$. [valid for $m = 0$ also].

7. HYPERBOLA

(8) Tangents to the Hyperbola from a point (x_1, y_1)

The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with slope m is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

If this tangent passes through (x_1, y_1), then

$$y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2}$$

$$\therefore (y_1 - mx_1)^2 = a^2m^2 - b^2$$

$$\therefore (x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0$$

This is a quadratic equation in m. So at the most two real values of m can be obtained. Thus, at the most two tangents can be drawn to the hyperbola from the point (x_1, y_1).

(9) Director Circle of Hyperbola

The equations of tangents to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with slope m are $y = mx \pm \sqrt{a^2m^2 - b^2}$.

If these tangents pass through the point $P(x_1, y_1)$, then

$$y_1 = mx_1 \pm \sqrt{a^2m^2 - b^2}.$$

$$\therefore (y_1 - mx_1)^2 = (a^2m^2 - b^2)$$

$$\therefore (x_1^2 - a^2)m^2 - 2x_1y_1m + y_1^2 + b^2 = 0.$$

If these two tangents are mutually perpendicular, then $m_1m_2 = -1$.

$$\therefore \quad \frac{y_1^2 + b^2}{x_1^2 - a^2} = -1 \qquad \therefore x_1^2 + y_1^2 = a^2 - b^2$$

Thus, the set of points of intersection of two mutually perpendicular tangents to the hyperbola are on the circle $x^2 + y^2 = a^2 - b^2$ called the Director Circle of hyperbola which does not exist for $a < b$ and is a point at the centre of the hyperbola for $a = b$.

(10) Asymptotes of a Hyperbola

If $\lim_{x\to\infty} |f(x) - mx - c| = 0$, then the line $y = mx + c$ is called an asymptote of the curve $y = f(x)$.
If as $x \to a$, $|f(x)| \to \infty$, then the line $x = a$ is called vertical asymptote of the curve $y = f(x)$.
The equation of the hyperbola is $x^2/a^2 - y^2/b^2 = 1$.
$\therefore\ y^2 = b^2(x^2/a^2 - 1) \quad \therefore\ y = f(x) = \pm\, b/a\,\sqrt{x^2 - a^2}$.
Suppose the line $y = mx + c$ is an asymptote of the hyperbola. Then $\lim_{x\to\infty} |f(x) - mx - c| = 0$.

$$\text{i.e., } \lim_{x\to\infty} \left| \pm \frac{b}{a}\sqrt{x^2 - a^2} - mx - c \right| = 0$$

$$\Rightarrow \lim_{x\to\infty} \left| x\left(\pm \frac{b}{a}\sqrt{1 - \frac{a^2}{x^2}} - m \right) - c \right| = 0$$

$$\Rightarrow c = 0 \text{ and } \lim_{x\to\infty} \pm \frac{b}{a}\sqrt{1 - \frac{a^2}{x^2}} - m = 0$$

$$\therefore\ c = 0 \text{ and } m = \pm \frac{b}{a}$$

$$\therefore \lim_{x\to\infty} \left| \pm \frac{b}{a}\sqrt{x^2 - a^2} - mx - c \right|$$

$$= \lim_{x\to\infty} \left| \pm \frac{b}{a}\left(\sqrt{x^2 - a^2} - x \right) \right|$$

$$= \lim_{x\to\infty} \left| \frac{b}{a} \right| \left| \frac{x^2 - a^2 - x^2}{\sqrt{x^2 - a^2} + x} \right|$$

$$= \lim_{x\to\infty} \frac{|ab|}{\sqrt{x^2 - a^2} + x} = 0.$$

Thus, the lines $y = \pm \frac{b}{a} x$ are the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$. The combined equations of the asymptotes $x/a - y/b = 0$ and $x/a + y/b = 0$ is $x^2/a^2 - y^2/b^2 = 0$. Thus, the equations of the hyperbola and its asymptotes differ only by a constant.

7. HYPERBOLA

(11) Rectangular Hyperbola

$\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$ are the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The slopes of these asymptotes are $m_1 = b/a$ and $m_2 = -b/a$.

If the angle between two asymptotes of a hyperbola is a right angle, then the hyperbola is called a rectangular hyperbola. Thus, for a rectangular hyperbola,

$$m_1 m_2 = -1 \Rightarrow (b/a)(-b/a) = -1 \Rightarrow b^2 = a^2.$$

$\therefore$ the equation of a rectangular hyperbola is $x^2 - y^2 = a^2$ and its eccentricity

$$e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{2a^2}{a^2}} = \sqrt{2}.$$

The parametric coordinates of any point on a rectangular hyperbola can be taken as $(a\sec\theta, a\tan\theta)$.

(12) Properties of a Hyperbola

(i) If P is any point on the hyperbola, whose foci are S and S′, then $|S'P - SP| = 2a$.

If P is any point of hyperbola and M is foot of perpendicular of P to directrix, then

$SP = e\,PM = e\left|x - \frac{a}{e}\right| = |ex - a|$

Similarly, $S'P = |ex + a|$

Now
$$\begin{aligned}(SP - S'P)^2 &= (SP)^2 - 2\,SP.S'P + (S'P)^2 \\ &= |ex-a|^2 - 2|e^2x^2 - a^2| + |ex+a|^2 \\ &= (e^2x^2 - 2exa + a^2) - 2|e^2x^2 - a^2| \\ &\quad + (e^2x^2 + 2exa + a^2)\end{aligned}$$

Now $e^2 > 1$ and $x^2 = a^2\sec^2\theta > a^2$

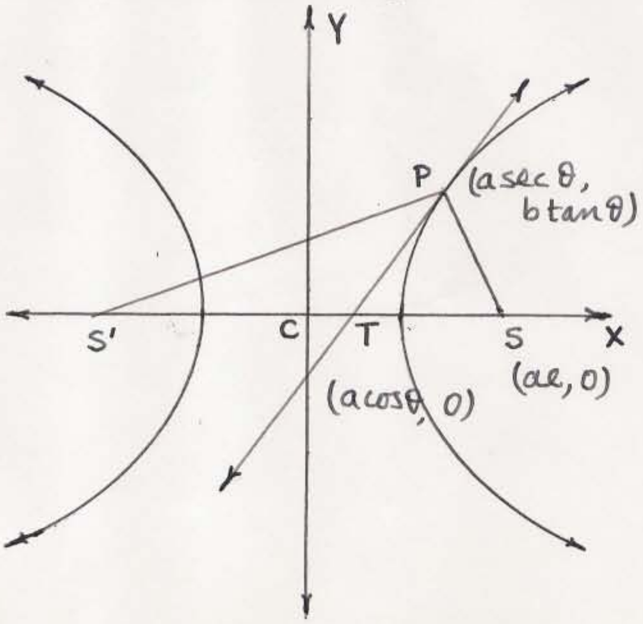
$\therefore\ e^2x^2 > a^2$

$\therefore\ e^2x^2 - a^2 > 0$

$\therefore\ (SP - S'P)^2 = 2(e^2x^2 + a^2) - 2(e^2x^2 - a^2) = 4a^2$

$\therefore\ |SP - S'P| = 2a.$

(ii) If P is any point on the hyperbola whose foci are S and S' and the tangent at P intersects the transverse axis at T, then $\dfrac{ST}{S'T} = \dfrac{SP}{S'P}$



The equation of the tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at $P(a\sec\theta, b\tan\theta)$ is $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1.$

Here, X-axis is the transverse axis of the hyperbola.

If this tangent intersects the X-axis in T, then the coordinates of T are $(a\cos\theta,\ 0)$

$\therefore\ ST^2 = (ae - a\cos\theta)^2 = a^2\cos^2\theta\,(e\sec\theta - 1)^2$

and $SP^2 = a^2(e\sec\theta - 1)^2$

$$\therefore\ ST^2 = SP^2\cos^2\theta \quad \text{... (i)}$$

similarly, $$S'T^2 = S'P^2\cos^2\theta \quad \text{... (ii)}$$

$\therefore\ \dfrac{SP}{S'P} = \dfrac{ST}{S'T}$

[NOTE: 'X-axis' in the text-book is changed to 'transverse axis' so that it applies to any hyperbola.]

(iii) If L and L' are the feet of perpendiculars from the foci S and S' of the hyperbola to the tangent at any point P, then $SL.S'L' = b^2$.

The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at any point $P(a\sec\theta, b\tan\theta)$ is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$.
The foci are $S(ae, 0)$ and $S'(-ae, 0)$.

$$\therefore \quad SL = \frac{|e\sec\theta - 1|}{\sqrt{\left(\frac{\sec\theta}{a}\right)^2 + \left(-\frac{\tan\theta}{b}\right)^2}} \quad \text{and}$$

$$S'L' = \frac{|e\sec\theta + 1|}{\sqrt{\left(\frac{\sec\theta}{a}\right)^2 + \left(-\frac{\tan\theta}{b}\right)^2}}$$

$$\therefore \quad SL.S'L' = \frac{e^2\sec^2\theta - 1}{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}$$

$$= \frac{\frac{a^2+b^2}{a^2}\sec^2\theta - 1}{\frac{b^2\sec^2\theta + a^2\tan^2\theta}{a^2b^2}} \quad \left[\begin{array}{l} \because b^2 = a^2(e^2-1) \\ \Rightarrow e^2 = \frac{a^2+b^2}{a^2} \end{array}\right]$$

$$= \frac{b^2(a^2\sec^2\theta + b^2\sec^2\theta - a^2)}{b^2\sec^2\theta + a^2\tan^2\theta}$$

$$= b^2 \quad (\because \sec^2\theta - 1 = \tan^2\theta).$$